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ON ONE GENERALIZATION OF THE HERMITE QUADRATURE FORMULA

Abstract. In this paper we propose a new approach to the construction of quadrature formulas of interpolation rational type on an interval. In the introduction, a brief analysis of the results on the topic of the research is carried out. Most attention is paid to the works of mathematicians of the Belarusian school on approximation theory – Gauss, Lobatto, and Radau quadrature formulas with nodes at the zeros of the rational Chebyshev – Markov fractions. Rational fractions on the segment, generalizing the classical orthogonal Jacobi polynomials with one weight, are defined, and some of their properties are described. One of the main results of this paper consists in constructing quadrature formulas with nodes at zeros of the introduced rational fractions, calculating their coefficients in an explicit form, and estimating the remainder. This result is preceded by some auxiliary statements describing the properties of special rational functions. Classical methods of mathematical analysis, approximation theory, and the theory of functions of a complex variable are used for proof. In the conclusion a numerical analysis of the efficiency of the constructed quadrature formulas is carried out. Meanwhile, the choice of the parameters on which the nodes of the quadrature formulas depend is made in several standard ways. The obtained results can be applied for further research of rational quadrature formulas, as well as in numerical analysis.

Keywords: approximation, interpolation, rational fractions, quadrature formulas, numerical analysis

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ОБ ОДНОМ ОБОБЩЕНИИ КВАДРАТУРНОЙ ФОРМУЛЫ ЭРМИТА

Аннотация. Целью данной работы является изучение нового подхода к построению квадратурных формул интерполяционно-рационального типа на отрезке. Проведен краткий анализ результатов по теме исследования, где основное внимание уделено работам математиков белорусской школы по теории аппроксимации – квадратурным формулам Гаусса, Лобатто, Радо с узлами в нулях рациональных дробей Чебышева – Маркова. Определяются рациональные дроби на отрезке, обобщающие классические ортогональные многочлены Якоби с одним весом, и описываются некоторые их свойства. Один из основных результатов работы состоит в построении квадратурных формул с узлами в нулях введенных рациональных дробей, вычислении их коэффициентов в явном виде, оценке остатка. Ему предшествуют некоторые вспомогательные утверждения, описывающие свойства специальных рациональных функций. Для доказательства используются классические методы математического анализа, теории приближений и теории функций комплексного переменного. Проводится численный анализ эффективности построенных квадратурных формул. При этом выбор параметров, от которых зависят узлы квадратурных формул, производится несколькими стандартными способами. Полученные результаты могут быть применены для дальнейшего исследования рациональных квадратурных формул, а также в численном анализе.

Ключевые слова: приближение, интерполяция, рациональные дроби, квадратурные формулы, численный анализ

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Introduction. Quadrature formulas based on rational interpolation are an intensively developing area of approximation theory. A large series of works are related to such studies (see, for example, [1–3] and the corresponding bibliography therein).

In the Belarusian school on the theory of rational approximation, the method of Gauss-type quadrature formulas, based on the interpolation at the zeroes of special algebraic Chebyshev – Markov fractions, has been developed. In 1996 quadrature formulas with nodes at the zeros of rational Chebyshev – Markov functions of the first and second kind were constructed (see [4]). Later, in [5], rational functions generalizing the Chebyshev polynomials of the third and fourth kind were introduced, and the corresponding quadrature formulas were constructed. Note that these quadrature formulas are exact for rational functions with poles of the second order. Finally, in [6] and [7], Lobatto and Radau quadrature formulas related to the extended system of Chebyshev – Markov knots of the second kind were constructed.

V. N. Rusak and his students I. V. Rybachenka and N. V. Grib in 2014–2015 (see [8, 9]) obtained a generalization of the quadrature formulas built in [4]. These formulas are exact for rational functions with simple poles.

In this paper, we find a generalization of the Gauss-type quadrature formula from [5] related to the weight $h(x)=\sqrt{(1-x)/(1+x)}$.

Main results. Let $n\in\mathbb{N}$ and $a_k,\ k=0,1,...,2n,$ be a set of numbers that satisfy the following conditions:

1) if $a_k\in\mathbb{R}$, then $|a_k|<1$;

2) if $a_k\in\mathbb{C}$, then among these numbers there is such a number a_l, that $a_l=\overline{a_k}$;

3) $a_0=a_{2n-1}=a_{2n}=0$.

Denote

$$\mu_{2n}(x)=\frac{1}{2}\sum_{k=0}^{2n}\arccos\frac{x+a_k}{1+a_kx}. \tag{1}$$

Notice that

$$\mu'_{2n}(x)=-\frac{\lambda_{2n}(x)}{\sqrt{1-x^2}}, \tag{2}$$

$$\lambda_{2n}(x)=\sum_{k=0}^{2n}\frac{\sqrt{1-a_k^2}}{1+a_kx}. \tag{3}$$

Let us define the algebraic fractions

$$Q_n(x)=\frac{\sin\mu_{2n}(x)}{\sqrt{1-x}},\quad x\in[-1;1]. \tag{4}$$

Note that the corresponding rational functions orthogonal on the interval [–1;1] were considered in [5] for the weight

$$h(x)=\sqrt{\frac{1-x}{1+x}}. \tag{5}$$

L e m m a 1. *For the function $Q_n(x)$ the following representation holds*

$$Q_n(x)=\frac{p_n(x)}{\sqrt{q_{2n-2}(x)}},$$

where $p_n(x)$ is an algebraic polynomial of the n degree,

$$q_{2n-2}(x)=\prod_{k=1}^{2n-2}(1+a_kx).$$

P r o o f. We use Euler's formula:

$$\sin\mu_{2n}(x)=\frac{1}{2i}\left(e^{i\mu_{2n}(x)}-e^{-i\mu_{2n}(x)}\right).$$

Taking into account definition (1) we have

$$e^{i\mu_{2n}(x)} = e^{i\frac{1}{2}\arccos x} e^{i\arccos x} \prod_{k=1}^{2n-2} e^{i\frac{1}{2}\arccos\frac{x+a_k}{1+a_kx}} =$$

$$= \frac{1}{2^{n-1/2}} \frac{\left(\sqrt{1+x}+i\sqrt{1-x}\right)\left(x+i\sqrt{1-x^2}\right)\prod_{k=1}^{2n-2}\left(\sqrt{(1+a_k)(1+x)}+i\sqrt{(1-a_k)(1-x)}\right)}{\prod_{k=1}^{2n-2}\sqrt{1+a_kx}}.$$

Hence,

$$\sin\mu_{2n}(x) = \operatorname{Im}\frac{1}{2^{n-1/2}} \frac{\left(\sqrt{1+x}+i\sqrt{1-x}\right)\left(x+i\sqrt{1-x^2}\right)\prod_{k=1}^{2n-2}\left(\sqrt{(1+a_k)(1+x)}+i\sqrt{(1-a_k)(1-x)}\right)}{\prod_{k=1}^{2n-2}\sqrt{1+a_kx}}.$$

It remains to note that

$$\prod_{k=1}^{2n-2}\left(\sqrt{(1+a_k)(1+x)}+i\sqrt{(1-a_k)(1-x)}\right) = u(x)+i\sqrt{1-x^2}\,v(x),$$

where $u(x)$ and $v(x)$ are the polynomials of the degree $n-1$ and $n-2$, respectively (see [10, p. 271]), and use definition (4). Lemma 1 is proved.

Since the function $\mu_{2n}(x)$ decreases monotonically over the segment [–1;1], the fraction $Q_n(x)$ has n simple zeroes on the interval (–1;1):

$$-1 < x_n < x_{n-1} < \ldots < x_1 < 1, \quad \mu_{2n}(x_k) = \pi k, \quad k = 1,2,\ldots,n.$$

Note, that (see (2) and (3))

$$Q_n'(x_k) = \left.\frac{\cos\mu_{2n}(x)\mu_{2n}'(x)\sqrt{1-x}+\sin\mu_{2n}(x)\dfrac{1}{2\sqrt{1-x}}}{\sqrt{1-x}}\right|_{x=x_k} =$$

$$= -\left.\frac{\cos\mu_{2n}(x)\lambda_{2n}(x)}{(1-x)\sqrt{1+x}}\right|_{x=x_k} = \frac{(-1)^{k+1}\lambda_{2n}(x_k)}{(1-x_k)\sqrt{1+x_k}}. \tag{6}$$

R e m a r k 1. The ratio between functions $\sin\mu_{2n}(x)$ and Bernstein rational fractions is well known (see [11, p. 49]). The following relation holds

$$\sin\mu_{2n}\left(\frac{1-y^2}{1+y^2}\right) = \sin\Phi_{2n}(y), \tag{7}$$

where $\sin\Phi_{2n}(y)$ is the Bernstein sine-fraction with zeroes at the points $\pm y_k$, $y_k = \sqrt{(1-x_k)/(1+x_k)},\ k = 1,2,\ldots,n.$ Besides

$$\sin\Phi_{2n}(y) = \frac{1}{2i}\left(\sqrt{\left(\frac{i-y}{i+y}\right)^3}\prod_{k=1}^{2n-2}\frac{z_k-y}{|z_k-y|} - \sqrt{\left(\frac{i+y}{i-y}\right)^3}\prod_{k=1}^{2n-2}\frac{\overline{z_k}-y}{|\overline{z_k}-y|}\right), \tag{8}$$

where points z_k are the roots of the equation $y^2+(1+a_k)/(1-a_k)=0,\ \operatorname{Im} z_k > 0,\ k=1,2,\ldots,2n-2.$ In this case, if a_p and a_q are complex conjugate, then z_p and z_q are symmetric with respect to the imaginary axis.

Using the definition of Bernstein fractions, it is easy to show that $(z_0 = z_{2n-1} = z_{2n} = i)$

$$\prod_{j=0}^{2n}\frac{z_j-y_k}{\overline{z_j}-y_k}=(-1)^k,\quad k=1,2,\ldots,n. \tag{9}$$

In addition, we note one more relation

$$\lambda_{2n}(x_k)=-\frac{i\left(1+y_k^2\right)}{2}\sum_{j=0}^{2n}\frac{z_j}{y_k^2-z_j^2}. \tag{10}$$

Now let us get back to the main result of our work. We define, as usual, the fundamental Lagrange polynomials

$$l_k(x)=\frac{Q_n(x)}{Q_n'(x_k)(x-x_k)},\quad k=1,2,\ldots,n, \tag{11}$$

and introduce the functions (see [12])

$$A_k(x)=\left(1-\frac{Q_n''(x_k)}{Q_n'(x_k)}(x-x_k)\right)l_k^2(x),\quad B_k(x)=(x-x_k)l_k^2(x),\quad k=1,2,\ldots,n. \tag{12}$$

By Lemma 1, $A_k(x)$ and $B_k(x)$ are rational functions of the form

$$\frac{p_{2n-1}(x)}{q_{2n-2}(x)}, \tag{13}$$

where $p_{2n-1}(x)$ is an algebraic polynomial of degree at most $2n-1$.

L e m m a 2. *For any rational function* $r_{2n-1}(x)$ *of form* (13) *the following representation holds*

$$r_{2n-1}(x)=\sum_{k=1}^{n}r_{2n-1}(x_k)A_k(x)+\sum_{k=1}^{n}r_{2n-1}'(x_k)B_k(x),\quad x\in\mathbb{R}. \tag{14}$$

P r o o f. Indeed, it is easy to check that for $k,m=1,2,\ldots,n$, the following equalities hold

$$A_k(x_m)=\begin{cases}0, & m\neq k,\\ 1, & m=k,\end{cases}\quad A_k'(x_m)=0;\quad B_k(x_m)=0,\quad B_k'(x_m)=\begin{cases}0, & m\neq k,\\ 1, & m=k.\end{cases} \tag{15}$$

We denote by $s_{2n-1}(x)$ the right-hand side of equality (14). Then using (15) we obtain

$$r_{2n-1}(x_m)-s_{2n-1}(x_m)=0,\quad r_{2n-1}'(x_m)-s_{2n-1}'(x_m)=0,\quad m=1,2,\ldots,n.$$

It remains to note that the denominators of the rational functions $r_{2n-1}(x)$ and $s_{2n-1}(x)$ coincide, and the numerators are algebraic polynomials of degree at most $2n-1$. Lemma 2 is proved.

C o r o l l a r y. *The following identity holds*

$$\sum_{k=1}^{n}A_k(x)\equiv 1. \tag{16}$$

P r o o f. Formula (16) follows immediately from (14) if we put $r_{2n-1}(x)\equiv 1$.

L e m m a 3. *Functions* $Q_n(x)$ *satisfy the following orthogonality conditions:*

$$\int_{-1}^{1}h(x)Q_n(x)\frac{x^k\,dx}{\sqrt{q_{2n-2}(x)}}=0,\quad k=0,1,\ldots,n-1,$$

where the weight function $h(x)$ *is defined by equality* (5).

P r o o f. Using definitions (4) and (5), we get

$$\int_{-1}^{1}h(x)Q_n(x)\frac{x^k\,dx}{\sqrt{q_{2n-2}(x)}}=\int_{-1}^{1}\frac{\sin\mu_{2n}(x)}{\sqrt{1+x}}\frac{x^k\,dx}{\sqrt{q_{2n-2}(x)}}:=I_n.$$

In the last integral we make substitution $x=(1-y^2)/(1+y^2)$. Then, taking into account relation (7), and the facts that

$$1+a_k x=1+a_k\frac{1-y^2}{1+y^2}=\frac{1-a_k}{1+y^2}\left(y^2+\frac{1+a_k}{1-a_k}\right)=\frac{1-a_k}{1+y^2}(z_k-y)(\overline{z_k}-y)$$

and

$$q_{2n-2}\left(\frac{1-y^2}{1+y^2}\right)=\frac{1}{(1+y^2)^{2n-2}}\prod_{k=1}^{2n-2}(1-a_k)\prod_{k=1}^{2n-2}(z_k-y)(\overline{z_k}-y),$$

we obtain

$$I_n=\frac{1}{\sqrt{2}\prod\limits_{k=1}^{2n-2}(1-a_k)}\int_{-\infty}^{+\infty}\left(\frac{(i-y)^2}{i+y}\prod_{k=1}^{2n-2}\frac{z_k-y}{|z_k-y|}-\frac{(i+y)^2}{i-y}\prod_{k=1}^{2n-2}\frac{\overline{z_k}-y}{|\overline{z_k}-y|}\right)\frac{(1-y^2)^k(1+y^2)^{n-3-k}\,ydy}{\sqrt{\prod\limits_{k=1}^{2n-2}(z_k-y)(\overline{z_k}-y)}}=$$

$$=\frac{1}{\sqrt{2}\prod\limits_{k=1}^{2n-2}(1-a_k)}\int_{-\infty}^{+\infty}\left(\frac{1}{(i+y)^3}\prod_{k=1}^{2n-2}\frac{1}{\overline{z_k}-y}-\frac{1}{(i-y)^3}\prod_{k=1}^{2n-2}\frac{1}{z_k-y}\right)(1-y^2)^k(1+y^2)^{n-1-k}\,ydy.$$

For $k=0,1,\ldots,n-1$ the integrand of the integral

$$\int_{-\infty}^{+\infty}\frac{(1-y^2)^k(1+y^2)^{n-1-k}}{(i+y)^3}\,y\prod_{k=1}^{2n-2}\frac{1}{\overline{z_k}-y}dy$$

has no singular points in the upper half-plane and it has infinity as a zero at least of the second order. Hence, this integral is equal to zero.

A similar statement is also true for the integral

$$\int_{-\infty}^{+\infty}\frac{(1-y^2)^k(1+y^2)^{n-1-k}}{(i-y)^3}\,y\prod_{k=1}^{2n-2}\frac{1}{z_k-y}dy.$$

Lemma 3 is proved. The next lemma describes some properties of the fundamental Lagrange polynomials (11).

L e m m a 4. *The following relations hold*

$$\int_{-1}^{1}h(x)l_k^2(x)dx=\pi\frac{1-x_k}{\lambda_{2n}(x_k)},\quad k=1,2,\ldots,n,$$

where the weight function $h(x)$ *is defined by equality* (5).

P r o o f. Using formulas (11) and (4), we get

$$\int_{-1}^{1}h(x)l_k^2(x)dx=\frac{1}{\left(Q_n'(x_k)\right)^2}\int_{-1}^{1}\frac{\sin^2\mu_{2n}(x)}{(x-x_k)^2}\frac{dx}{\sqrt{1-x^2}}:=\frac{1}{\left(Q_n'(x_k)\right)^2}I_k,\quad k=1,2,\ldots,n. \tag{17}$$

We substitute $x=(1-y^2)/(1+y^2)$. In this case (see also remark 1)

$$x-x_k=\frac{1-y^2}{1+y^2}-\frac{1-y_k^2}{1+y_k^2}=\frac{1+y_k^2-y^2-y^2y_k^2-1-y^2+y_k^2+y^2y_k^2}{(1+y^2)(1+y_k^2)}=-2\frac{y^2-y_k^2}{(1+y^2)(1+y_k^2)},$$

where y_k, $k=1,2,\ldots,n$, are the zeroes of Bernstein sine-fraction (8). Besides,

$$1-x^2{}_k=1-\frac{(1-y^2)^2}{(1+y^2)^2}=\frac{(1+y^2)^2-(1-y^2)^2}{(1+y^2)^2}=\frac{4y^2}{(1+y^2)^2},\quad dx=\frac{-4ydy}{(1+y^2)^2}.$$

Therefore, we obtain

$$I_k=\int\limits_{+\infty}^{0}\frac{\sin^2\Phi_{2n}(y)}{4\dfrac{\left(y^2-y_k^2\right)^2}{(1+y^2)^2\left(1+y_k^2\right)^2}}\frac{1+y^2}{2y}\frac{-4ydy}{(1+y^2)^2}=\frac{\left(1+y_k^2\right)^2}{4}\int\limits_{-\infty}^{+\infty}\sin^2\Phi_{2n}(y)\frac{1+y^2}{\left(y^2-y_k^2\right)^2}dy. \quad (18)$$

To calculate the last integral, we use the method proposed in [13]. For this purpose, we consider the integral

$$J_n(z):=\int\limits_{-\infty}^{+\infty}\sin^2\Phi_{2n}(y)\frac{1+y^2}{(y^2-z^2)^2}dy,\qquad \operatorname{Im} z>0.$$

Taking into account representation (8), we have the equality

$$\sin^2\Phi_{2n}(y)=-\frac{1}{4}\left(\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}-2+\prod_{j=0}^{2n}\frac{\overline{z_j}-y}{z_j-y}\right).$$

Hence

$$J_n(z)=-\frac{1}{4}\left(J_n^{(1)}(z)-2J_n^{(2)}(z)+J_n^{(3)}(z)\right), \quad (19)$$

where

$$J_n^{(1)}(z)=\int\limits_{-\infty}^{+\infty}\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}\frac{1+y^2}{(y^2-z^2)^2}dy,\quad J_n^{(2)}(z)=\int\limits_{-\infty}^{+\infty}\frac{1+y^2}{(y^2-z^2)^2}dy,\quad J_n^{(3)}(z)=\int\limits_{-\infty}^{+\infty}\prod_{j=0}^{2n}\frac{\overline{z_j}-y}{z_j-y}\frac{1+y^2}{(y^2-z^2)^2}dy.$$

Consider the integral $J_n^{(1)}(z)$. In the upper half-plane its integrand has $y=z$ as the only pole of the second order. Therefore

$$J_n^{(1)}(z)=2\pi i\operatorname*{res}_{y=z}\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}\frac{1+y^2}{(y^2-z^2)^2}=2\pi i\lim_{y\to z}\frac{d}{dy}\left(\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}\frac{1+y^2}{(y+z)^2}\right).$$

Notice, that

$$\frac{d}{dy}\left(\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}\right)=\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}\sum_{j=0}^{2n}\frac{\overline{z_j}-y}{z_j-y}\frac{d}{dy}\left(\frac{z_j-y}{\overline{z_j}-y}\right)=$$

$$=\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-y)(\overline{z_j}-y)},\qquad \frac{d}{dy}\left(\frac{1+y^2}{(y+z)^2}\right)=2\frac{yz-1}{(y+z)^3}.$$

Then

$$J_n^{(1)}(z)=2\pi i\lim_{y\to z}\left(\frac{1+y^2}{(y+z)^2}\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-y)(\overline{z_j}-y)}+2\frac{yz-1}{(y+z)^3}\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}\right)=$$

$$=2\pi i\lim_{y\to z}\frac{1+y^2}{(y+z)^2}\prod_{j=0}^{2n}\frac{z_j-y}{\overline{z_j}-y}\left(\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-y)(\overline{z_j}-y)}+2\frac{yz-1}{(1+y^2)(y+z)}\right)=$$

$$=\frac{\pi i}{2}\frac{1+z^2}{z^2}\prod_{j=0}^{2n}\frac{z_j-z}{\overline{z_j}-z}\left(\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-z)(\overline{z_j}-z)}+\frac{z^2-1}{(1+z^2)z}\right). \quad (20)$$

Similar reasoning for the integral $J_n^{(2)}(z)$ leads to the fact that

$$J_n^{(2)}(z)=2\pi i \operatorname*{res}_{y=z}\frac{1+y^2}{(y^2-z^2)^2}=2\pi i\lim_{y\to z}\frac{d}{dy}\left(\frac{1+y^2}{(y+z)^2}\right)=2\pi i\lim_{y\to z}\frac{2yz-2}{(y+z)^3}=\frac{\pi i}{2}\frac{z^2-1}{z^3}. \qquad (21)$$

It remains to calculate the integral

$$J_n^{(3)}(z)=\int_{-\infty}^{+\infty}\prod_{j=0}^{2n}\frac{\overline{z_j}-y}{z_j-y}\frac{1+y^2}{(y^2-z^2)^2}dy.$$

Now we will integrate over the lower half-plane. We have

$$J_n^{(3)}(z)=-2\pi i \operatorname*{res}_{y=-z}\prod_{j=0}^{2n}\frac{\overline{z_j}-y}{z_j-y}\frac{1+y^2}{(y^2-z^2)^2}=-2\pi i\lim_{y\to -z}\frac{d}{dy}\left(\prod_{j=0}^{2n}\frac{\overline{z_j}-y}{z_j-y}\frac{1+y^2}{(y^2-z^2)^2}\right)=$$

$$=-2\pi i\lim_{y\to -z}\left(\frac{1+y^2}{(y-z)^2}\prod_{j=0}^{2n}\frac{\overline{z_j}-y}{z_j-y}\sum_{j=0}^{2n}\frac{\overline{z_j}-z_j}{(z_j-y)(\overline{z_j}-y)}-2\frac{yz+1}{(y-z)^3}\prod_{j=0}^{2n}\frac{\overline{z_j}-y}{z_j-y}\right)=$$

$$=-2\pi i\lim_{y\to -z}\frac{1+y^2}{(y-z)^2}\prod_{j=0}^{2n}\frac{\overline{z_j}-y}{z_j-y}\left(\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-y)(\overline{z_j}-y)}-2\frac{yz+1}{(1+y^2)(y-z)}\right)=$$

$$=-\frac{\pi i}{2}\frac{1+z^2}{z^2}\prod_{j=0}^{2n}\frac{\overline{z_j}+y}{z_j+y}\left(\sum_{j=0}^{2n}\frac{\overline{z_j}-z_j}{(z_j+z)(\overline{z_j}+z)}-\frac{z^2-1}{(1+z^2)z}\right).$$

Since the numbers z_k, $k=0,1,...,2n$, are symmetric with respect to the imaginary axis, we obtain

$$\prod_{j=0}^{2n}\frac{\overline{z_j}+y}{z_j+y}=\prod_{j=0}^{2n}\frac{z_j-z}{\overline{z_j}-z},\qquad \sum_{j=0}^{2n}\frac{\overline{z_j}-z_j}{(z_j+z)(\overline{z_j}+z)}=-\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-z)(\overline{z_j}-z)}.$$

Then, taking into account (20), we get

$$J_n^{(3)}(z)=\frac{\pi i}{2}\frac{1+z^2}{z^2}\prod_{j=0}^{2n}\frac{z_j-z}{\overline{z_j}-z}\left(\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-z)(\overline{z_j}-z)}+\frac{z^2-1}{(1+z^2)z}\right)=J_n^{(1)}(z). \qquad (22)$$

Substituting (20), (21), and (22) into (19), we obtain

$$J_n(z)=-\frac{\pi i}{4}\left(\frac{1+z^2}{z^2}\prod_{j=0}^{2n}\frac{z_j-z}{\overline{z_j}-z}\left(\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-z)(\overline{z_j}-z)}+\frac{z^2-1}{(1+z^2)z}\right)-\frac{z^2-1}{z^3}\right).$$

To find integral (18), we pass to the limit

$$I_k=\frac{\left(1+y_k^2\right)^2}{4}\lim_{z\to y_k,\operatorname{Im}z>0}J_n(z)=$$

$$=-\frac{\pi i\left(1+y_k^2\right)^2}{16}\left(\frac{1+y_k^2}{y_k^2}\prod_{j=0}^{2n}\frac{z_j-y_k}{\overline{z_j}-y_k}\left(\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-y_k)(\overline{z_j}-y_k)}+\frac{y_k^2-1}{\left(1+y_k^2\right)y_k}\right)-\frac{y_k^2-1}{y_k^3}\right).$$

Due to (9) we get

$$I_k=-\frac{\pi i\left(1+y_k^2\right)^2}{16}\left(\frac{1+y_k^2}{y_k^2}\left(\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-y_k)(\overline{z_j}-y_k)}+\frac{y_k^2-1}{\left(1+y_k^2\right)y_k}\right)-\frac{y_k^2-1}{y_k^3}\right)=$$

$$=-\frac{\pi i\left(1+y_k^2\right)^2}{16y_k^2}\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-y_k)(\overline{z_j}-y_k)}.$$

Now we note that, since the parameters z_k, $k=0,1,...,2n$, are symmetric, the following equality holds

$$\sum_{j=0}^{2n}\frac{z_j-\overline{z_j}}{(z_j-y_k)(\overline{z_j}-y_k)}=\sum_{j=0}^{2n}\left(\frac{1}{y_k-z_j}-\frac{1}{y_k-\overline{z_j}}\right)=2\sum_{j=0}^{2n}\frac{z_j}{y_k^2-z_j^2}.$$

Therefore, using relation (20), we obtain

$$I_k=-\frac{\pi i\left(1+y_k^2\right)^2}{16y_k^2}\left(-\frac{4\lambda_{2n}(x_k)}{i\left(1+y_k^2\right)}\right)=\frac{\pi i\left(1+y_k^2\right)^2\lambda_{2n}(x_k)}{4y_k^2}=\frac{\pi\lambda_{2n}(x_k)}{1-x_k^2},\quad k=1,2,...,n.$$

Finally, we substitute the obtained relations into (17) and take into account (6)

$$\int_{-1}^{1}h(x)l_k^2(x)dx=\frac{(1-x_k)^2(1+x_k)\pi\lambda_{2n}(x_k)}{\lambda_{2n}^2(x_k)}\frac{\pi\lambda_{2n}(x_k)}{1-x_k^2}=\pi\frac{1-x_k}{\lambda_{2n}(x_k)},\quad k=1,2,...,n.$$

Lemma 4 is proved.

Let $f(x)$ be an integrable function with respect to weight (6) on the segment [–1;1]. Denote

$$C_k=\int_{-1}^{1}h(x)l_k^2(x)dx,\quad k=1,2,...,n,$$

and consider the quadrature formula

$$\int_{-1}^{1}h(x)f(x)dx=\sum_{k=1}^{n}C_kf(x_k)+\rho_n(f). \tag{23}$$

T h e o r e m. *Quadrature formula* (23) *can be expressed as*

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)dx=\pi\sum_{k=1}^{n}\frac{1-x_k}{\lambda_{2n}(x_k)}f(x_k)+\rho_n(f). \tag{24}$$

This quadrature formula is exact for any function of form (13) *and for any function* $f(x)$ *continuous on* [–1;1] *the following estimation of its remainder holds*

$$|\rho_n(f)|\le 2\pi R_{2n-1}(f,a),$$

where $R_{2n-1}(f,a)$ *is the best uniform approximation of a function* $f(x)$ *by rational functions of form* (13) *of order at most* $2n-1$.

P r o o f. Let us consider a rational function $r_{2n-1}(x)$ of form (13). According to lemma 2, equality (14) holds. We multiply both sides of this equality by weight function $h(x)$ and integrate over segment [–1;1]:

$$\int_{-1}^{1}h(x)r_{2n-1}(x)dx=\sum_{k=1}^{n}r_{2n-1}(x_k)\int_{-1}^{1}h(x)A_k(x)dx+\sum_{k=1}^{n}r'_{2n-1}(x_k)\int_{-1}^{1}h(x)B_k(x)dx. \tag{25}$$

Notice, that due to (12) and (11)

$$\int_{-1}^{1}h(x)A_k(x)dx=\int_{-1}^{1}h(x)\left(1-\frac{Q_n''(x_k)}{Q_n'(x_k)}(x-x_k)\right)l_k^2(x)dx=$$
$$=\int_{-1}^{1}h(x)l_k^2(x)dx-\frac{Q_n''(x_k)}{(Q_n'(x_k))^3}\int_{-1}^{1}h(x)Q_n(x)\frac{Q_n(x)}{x-x_k}dx.$$

Using the fact, that $Q_n(x_k)=0,\ \ k=1,2,...,n$, lemmas 1 and 3, we get

$$\int_{-1}^{1}h(x)Q_n(x)\frac{Q_n(x)}{x-x_k}dx=0,\quad k=1,2,...,n.$$

It means

$$\int_{-1}^{1} h(x)A_k(x)dx = \int_{-1}^{1} h(x)l_k^2(x)dx.$$

Similar reason leads to the fact that

$$\int_{-1}^{1} h(x)B_k(x)dx = 0, \quad k = 1,2,...,n.$$

Therefore, taking into account equality (25) and the result of lemma 4, we have

$$\int_{-1}^{1} h(x)r_{2n-1}(x)dx = \int_{-1}^{1} h(x)l_k^2(x)dx = \sum_{k=1}^{n} C_k r_{2n-1}(x_k).$$

Thus, the exactness of formula (23) on rational functions of form (13) is proved. If $r_{2n-1}(x) \equiv 1$ in the last equality, then

$$\sum_{k=1}^{n} C_k = \int_{-1}^{1} h(x)dx = \pi. \tag{26}$$

Now let $f(x) \in C[-1,1]$ and $r_{2n-1}^{*}(x)$ be a rational function of the best uniform approximation of the function $f(x)$ by functions of form (13). Then, taking into account equality (26) and the positiveness of the coefficients C_k, $k = 1,2,...,n$, we obtain the estimation

$$|\rho_n(f)| = \left| \int_{-1}^{1} h(x)f(x)dx - \sum_{k=1}^{n} C_k f(x_k) \right| \le \int_{-1}^{1} h(x)\,|\,f(x) - r_{2n-1}^{*}(x)\,|\,dx +$$
$$+ \sum_{k=1}^{n} C_k\,|\,f(x_k) - r_{2n-1}^{*}(x_k)\,| \le 2\pi R_{2n-1}(f,a).$$

This concludes the proof of theorem 1.

R e m a r k 2. If in formula (24) $a_k = 0$, $k = 0,1,...,2n$, taking into account definition (3) we obtain the classical polynomial-type quadrature formula (see [14, p. 618]). It is natural to call it a Hermite quadrature formula.

E x a m p l e. It is interesting to illustrate numerically the application of the obtained quadrature formula (24) for calculating definite integrals. Taking into account the way of constructing this formula, we choose a function with a singularity near a point $x = 1$ as a function $f(x)$.

Consider the integral

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \frac{1}{\sqrt[3]{1-x}} dx, \quad \text{i. e.} \quad f(x) = \frac{1}{\sqrt[3]{1-x}}$$

(see (24)). We will carry out calculations in the following three situations:

1) $a_k = 0$, $k = 0,1,...,2n$ (polynomial case);

2) $a_0 = a_{2n-1} = a_{2n} = 0$, $a_k = -0.99$, $k = 1,2,...,2n-2$;

3) $a_0 = a_{2n-1} = a_{2n} = 0$, $a_k = -1 + 1/n$, $k = 1,2,...,2n-2$.

The numerical procedure was done with the help of Python 3.7.2 and its results are presented in Table.

n	Case 1		Case 2		Case 3	
	Integral	Error	Integral	Error	Integral	Error
10	2.8911147109	0.0003172341	2.8914206929	0.0000112521	2.8914268291	0.0000051159
20	2.8913671788	0.0000647662	2.8914317894	0.0000001556	2.8914315173	0.0000004277
30	2.8914065566	0.0000253884	2.8914318892	0.0000000558	2.8914318436	0.0000001014

End of Table

n	Case 1		Case 2		Case 3	
	Integral	Error	Integral	Error	Integral	Error
40	2.8914189072	0.0000130378	2.8914319167	0.0000000283	2.8914319083	0.0000000367
50	2.8914241765	0.0000077685	2.8914319283	0.0000000167	2.8914319283	0.0000000167
60	2.8914268585	0.0000050865	2.8914319341	0.0000000109	2.8914319362	0.0000000088
70	2.8914283902	0.0000035548	2.8914319374	0.0000000076	2.8914319399	0.0000000051
80	2.8914293391	0.0000026059	2.8914319395	0.0000000055	2.8914319418	0.0000000032
90	2.8914299637	0.0000019813	2.8914319408	0.0000000042	2.8914319429	0.0000000021
100	2.8914303945	0.0000015505	2.8914319417	0.0000000033	2.8914319435	0.0000000015

Analysing the results, one can conclude the following. Calculation using the obtained quadrature formula in the rational case even with the simplest choice of parameters $\{a_k\}_{k=0}^{2n}$ is much more efficient than in the corresponding polynomial case. Making the choice of parameters more complex can lead to improving the results.

Conclusion. In the present paper a new approach to the construction of an interpolation rational quadrature formula of Gauss type on an interval is described. For this purpose, the properties of special rational functions and the fundamental Lagrange polynomials, constructed on their basis, are studied. The coefficients of the quadrature formula are obtained in an explicit form, and the remainder is estimated. A numerical analysis of the efficiency of the constructed quadrature formula is carried out.

References

1. Deckers K., Mougaida A., Belhadjsalah H. Algorithm 973: extended rational Fejér quadrature rules based on Chebyshev orthogonal rational functions. *ACM Transactions on Mathematical Software*, 2017, vol. 43, no. 4, pp. 15–66. https://doi.org/10.1145/3054077

2. Deckers K. Christoffel–Darboux-type formulae for orthonormal rational functions with arbitrary complex poles. *IMA Journal of Numerical Analysis*, 2015, vol. 35, no. 4, pp. 1842–1863. https://doi.org/10.1093/imanum/dru049

3. Deckers K., Bultheel A., Perdomo-Pío F. Rational Gauss-Radau and Szegö-Lobatto quadrature on the interval and the unit circle respectively. *Jaen Journal on Approximation*, 2011, vol. 3, no. 1, pp. 15–66.

4. Rouba Y. A. Quadrature formulas of interpolation rational type. *Doklady Natsional'noi akademii nauk Belarusi = Doklady of the National Academy of Sciences of Belarus*, 1996, vol. 40, no. 3, pp. 42–46 (in Russian).

5. Rouba Y. A. On one orthogonal system of rational functions and quadratures of Gauss type. *Vestsi Natsyianal'nai akademii navuk Belarusi. Seryia fizika-matematychnykh navuk = Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series*, 1998, no. 3, pp. 31–35 (in Russian).

6. Rouba Y. A., Dirvuk Y. V. On one quadrature formula of interpolation rational type with respect to Chebyshev – Markov nodes. *Doklady Natsional'noi akademii nauk Belarusi = Doklady of the National Academy of Sciences of Belarus*, 2014, vol. 58, no. 5, pp. 23–29 (in Russian).

7. Dirvuk Y. V., Smatrytski K. A. Rational quadrature formulas of Radau type. *Vestnik BGU. Seriya 1, Fizika. Matematika. Informatika = Proceedings of BSU. Series 1. Physics. Mathematics. Informatics*, 2014, no. 1, pp. 87–91 (in Russian).

8. Rusak V. N., Rybachenka I. V. Chebyshev – Markov's cosine-fractions in the approximate integration. *Vestsi Natsyianal'nai akademii navuk Belarusi. Seryia fizika-matematychnykh navuk = Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series*, 2014, no. 3, pp. 15–20 (in Russian).

9. Rusak V. N., Grib N. V. Chebyshev – Markov sine-fractions in approximate integration. *Vestsi BDPU. Seryja 3. Fizika. Matematyka. Infarmatyka. Biyalogiya. Geagrafiya. = Proceedings of BSPU. Series 3. Physics. Mathematics. Informatics. Biology. Geography*, 2015, no. 2, pp. 17–20 (in Russian).

10. Markov A. A. *Selected Works on the Theory of Continuous Fractions and the Theory of Functions with the Least Deviation from Zero*. Moscow, Leningrad, GITTL Publ., 1948. 413 p. (in Russian).

11. Rusak V. N. *Rational Functions as Approximation Apparatus*. Minsk, BSU Publ., 1979. 176 p. (in Russian).

12. Szabados J., Vertesi P. *Interpolation of Functions*. World Scientific, 1990. 305 p.

13. Rouba Y. A., Smatrytski K. A. Convergence in the mean of rational interpolating processes in the zeroes of Bernstein fractures. *Vestsi Natsyianal'nai akademii navuk Belarusi. Seryia fizika-matematychnykh navuk = Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series*, 2010, no. 3, pp. 5–9 (in Russian).

14. Natanson I. P. *Constructive Theory of Functions.* Moscow, Leningrad, Gostechizdat Publ., 1949. 688 p. (in Russian).

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