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A SPIN 1 PARTICLE IN A CYLINDRIC BASIS: THE PROJECTIVE OPERATOR METHOD

Abstract. In this paper, the system of equations describing a spin 1 particle is studied in cylindric coordinates with the use of tetrad formalism and the matrix 10-dimension formalism of Duffin – Kemmer – Petieau. After separating the variables, we apply the method proposed by Fedorov – Gronskiy and based on the use of projective operators to resolve the system of 10 equations in the *r* variable. In the presence of an external uniform magnetic field, we construct in an explicit form three independent classes of wave functions with corresponding energy spectra. Separately the massless field with spin 1 is studied; there are found four linearly independent solutions, two of which are gauge ones, and other two do not contain gauge degrees of freedom. Meanwhile, the method of Fedorov – Gronskiy is also used.

Keywords: spin 1 field, Duffin – Kemmer equation, cylindrical symmetry, method of projective operators, massive and massless particles, gauge degrees of freedom

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ЧАСТИЦА СО СПИНОМ 1 В ЦИЛИНДРИЧЕСКОМ БАЗИСЕ: МЕТОД ПРОЕКТИВНЫХ ОПЕРАТОРОВ

Аннотация. В настоящей работе система уравнений, описывающая частицу со спином 1, изучается в цилиндрических координатах с использованием тетрадного формализма и матричного 10-мерного формализма Даффина – Кеммера – Петье. После разделения переменных для решения системы 10 уравнений относительно переменной применяется метод, предложенный Федоровым – Гронским и основанный на применении проективных операторов. При наличии внешнего однородного магнитного поля построены в явном виде три независимых класса волновых функций с соответствующими энергетическими спектрами. Отдельно исследуется безмассовое поле со спином 1; найдено четыре линейно независимых решения, два из которых калибровочные, а остальные два не содержат калибровочных степеней свободы. При этом также используется метод Федорова – Гронского.

Ключевые слова: поле со спином 1, уравнение Даффина – Кеммера, цилиндрическая симметрия, метод проективных операторов, массивная и безмассовые частицы, калибровочные степени свободы

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Introduction, initial equations. In this paper, the system of equations describing a spin 1 particle is studied in cylindric coordinates with the use of tetrad formalism and matrix 10-dimension formalism of Duffin – Kemmer – Petieau. Applying the Fedorov – Gronskiy method [1, 2], we examine this field in the massive case in the presence of an external magnetic field, getting three independent classes of wave functions and corresponding energy spectra. We also study the massless particle, focusing attention on separating two gauge solutions. At this also the Fedorov – Gronskiy method is used. The Proca system of tensor equations for the vector particle has the form

$$D^b \Psi_{ab} - M\Psi_a = 0, \quad D_a \Psi_b - D_b \Psi_a - M\Psi_{ab} = 0, \tag{1}$$

where $D_a = \partial_a + ieA_a$. We will use the wave function

$$\Phi = (\Psi_0, \Psi_1, \Psi_2, \Psi_3; \Psi_{01}, \Psi_{02}, \Psi_{03}, \Psi_{23}, \Psi_{31}, \Psi_{12}) = (H_1; H_2).$$

Let us transform eqs. (1) to the matrix form. The first equation gives $K^a D_a H_2 - MH_1 = 0$, where

$$K^0 = \begin{vmatrix} . & . & . & . & . & . \\ -1 & . & . & . & . & . \\ . & -1 & . & . & . & . \\ . & . & -1 & . & . & . \end{vmatrix}, K^1 = \begin{vmatrix} -1 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 1 \\ . & . & . & . & -1 & . \end{vmatrix}, K^2 = \begin{vmatrix} . & -1 & . & . & . & . \\ . & . & . & . & . & -1 \\ . & . & . & . & . & . \\ . & . & . & +1 & . & . \end{vmatrix}, K^3 = \begin{vmatrix} . & . & -1 & . & . & . \\ . & . & . & . & +1 & . \\ . & . & . & -1 & . & . \\ . & . & . & . & . & . \end{vmatrix}.$$

The second equation leads to $D_a L^a H_1 - MH_2 = 0$, where

$$L^0 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad L^1 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad L^2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad L^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

So the system (1) can be written as follows

$$K^a D_a H_2 - MH_1 = 0, \quad D_a L^a H_1 - MH_2 = 0,$$
$$\left(D_a \beta^a - M\right)\Phi = 0, \quad \beta^a = \begin{vmatrix} 0 & K^a \\ L^a & 0 \end{vmatrix}, \quad \Phi = \begin{vmatrix} H_1 \\ H_2 \end{vmatrix}. \tag{2}$$

Let us generalize this equation to the Riemannian space-time. For any given metric $g_{\alpha\beta}(x)$, we choose some tetrad $dS^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta$, $g_{\alpha\beta}(x) \to e_{(a)\alpha}(x)$; then the covariant equation should be written as [3, 4]

$$\left[\beta^\alpha(x)\left(\frac{\partial}{\partial x^\alpha} + \Sigma_\alpha(x)\right) - M\right]\Psi(x) = 0, \tag{3}$$

where the local matrices $\beta^\alpha(x)$ and their blocks are defined with the help of the tetrads

$$\beta^\alpha(x) = e^\alpha_{(a)}(x)\beta^a = \begin{vmatrix} 0 & K^a e^\alpha_{(a)} \\ L^a e^\alpha_{(a)} & 0 \end{vmatrix}. \tag{4}$$

The corresponding connection $\Sigma_\alpha(x)$ is defined by the following relations

$$J^{ab} = \begin{vmatrix} J_1^{ab} & 0 \\ 0 & J_2^{ab} \end{vmatrix}, \ \Sigma_\alpha(x) = \frac{1}{2} J^{ab} e^\beta_{(a)}(x)\, e_{(b)\beta;\alpha}(x) = \begin{vmatrix} (\Sigma_1)_\alpha & 0 \\ 0 & (\Sigma_2)_\alpha \end{vmatrix}, \tag{5}$$

symbols $J^{ab}_{(1)}$ and $J^{ab}_{(2)}$ stand for generators for the vector $\Psi_k(x)$ and the antisymmetric tensor $\Psi_{mn}(x)$. Equation (3) may be presented with the use of the Ricci rotation coefficients

$$\left[\beta^c\left(e^\alpha_{(c)}\frac{\partial}{\partial x^\alpha} + ieA_c + \frac{1}{2}J^{ab}\gamma_{abc}\right) - M\right]\Psi(x) = 0, \quad \gamma_{[ab]c} = -\gamma_{[ba]c} = e_{(b)\rho;\sigma}e^\rho_{(a)}e^\sigma_{(c)}. \tag{6}$$

In the block form, instead of (6) we have equations

$$\begin{gathered} \left[K^c e^\alpha_{(c)}\partial_\alpha + K^c J^{ab}_{(2)}\frac{1}{2}\gamma_{abc}\right]H_2 - MH_1 = 0, \\ \left[L^c e^\alpha_{(c)}\partial_\alpha + L^c J^{ab}_{(1)}\frac{1}{2}\gamma_{abc}\right]H_1 - MH_2 = 0. \end{gathered} \tag{7}$$

Cylindric coordinates and tetrad. In cylindric coordinates $x^\alpha = (t, r, \phi, z)$ and tetrad (see [4, 5]), equations (7) can be written in the form (we take into account the presence of the external uniform magnetic field)

$$\begin{gathered} \left[K^0\frac{\partial}{\partial t} + K^1\frac{\partial}{\partial r} + K^2\frac{\partial_\phi + eBr^2/2 + J^{12}_{(2)}}{r} + K^3\frac{\partial}{\partial z}\right]H_2 = MH_1, \\ \left[L^0\frac{\partial}{\partial t} + L^1\frac{\partial}{\partial r} + L^2\frac{\partial_\phi + eBr^2/2 + J^{12}_{(1)}}{r} + L^3\frac{\partial}{\partial z}\right]H_1 = MH_2. \end{gathered} \tag{8}$$

When separating the variables, it is convenient to use the cyclic basis. This basis is defined by the requirement of diagonality of the matrix j^{12} for the vector field $H_1 = (\Psi_l)$. The necessary transformation $\bar{H}_1 = UH_1$ is determined by the following matrix U:

$$U = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} & 0 \end{vmatrix}; \tag{9}$$

we can verify that the needed equality is valid. Correspondingly, the generators for the vector and tensor transform to the cyclic basis in accordance with the rules

$$\bar{J}_1^{ab} = Uj^{ab}U^{-1}, \quad \bar{J}_2^{ab} = \bar{j}^{ab}\otimes I + I\otimes\bar{j}^{ab}. \tag{10}$$

Let us find the 6-dimensional form of the tensor generator $J^{12}_{(2)}$ (firstly in the Cartesian basis):

$$J_2^{12}H_2 = j^{12}H_2 + H_2\tilde{j}^{12}, \quad H_2 = \{E_i, B_i\} \ \Rightarrow \ J^{12}_{(2)} = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}. \tag{11}$$

Let us find the cyclic representation for this generator; the initial formula has the form

$$\Phi_{[kl]} = \varphi = \begin{vmatrix} 0 & E_{10} & E_{20} & E_{30} \\ -E_{10} & 0 & B_{30} & -B_{20} \\ -E_{20} & -B_{30} & 0 & B_{10} \\ -E_{30} & B_{20} & -B_{10} & 0 \end{vmatrix}, \quad \bar{J}^{12}_{(2)}\varphi = \bar{j}^{12}\varphi + \varphi\tilde{\bar{j}}^{12};$$

with the use of which we derive

$$\bar{J}^{12}_{(2)}\varphi = i\begin{vmatrix} 0 & -E_{10} & 0 & E_{30} \\ E_{10} & 0 & -B_{30} & 0 \\ 0 & B_{30} & 0 & B_{10} \\ -E_{30} & 0 & -B_{10} & 0 \end{vmatrix}, \quad \bar{J}^{12}_{2} = i\begin{vmatrix} -1 & . & . & . & . & . \\ . & 0 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 0 & . \\ . & . & . & . & . & -1 \end{vmatrix}. \tag{12}$$

It is also necessary to transform the matrices of equation (6) to the cyclic basis. We will perform the transformation in the block form: $\bar{H}_1 = UH_1, \bar{H}_2 = (U \otimes U)H_2 = C_2H_2$; so we get the rule

$$\begin{vmatrix} 0 & \bar{K}^a \\ \bar{L}^a & 0 \end{vmatrix} = \begin{vmatrix} 0 & C_1K^aC_2^{-1} \\ C_2L^aC_1^{-1} & 0 \end{vmatrix}.$$

We find the explicit form of the matrix C_2 in $\bar{H}_2 = C_2H_2$, and it is omitted. Further, we obtain expressions for the matrix blocks in the cyclic basis, and they are also omitted.

We separate the variables in equations (8) taking into account the substitutions

$$\bar{H}_1 = e^{-i\epsilon t}e^{im\phi}e^{ikz}\begin{vmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{vmatrix}, \quad \bar{H}_2 = e^{-i\epsilon t}e^{im\phi}e^{ikz}\begin{vmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix}. \tag{13}$$

With the use of notations

$$a_m = \frac{d}{dr} + \frac{m + eBr^2/2}{r}, \quad a_{m+1} = \frac{d}{dr} + \frac{m + 1 + eBr^2/2}{r},$$
$$b_m = \frac{d}{dr} - \frac{m + eBr^2/2}{r}, \quad b_{m-1} = \frac{d}{dr} - \frac{m - 1 + eBr^2/2}{r},$$

the resulting system of differential equations in the variable r reads as follows

$$\begin{gathered} b_{m-1}E_1 - a_{m+1}E_3 - \sqrt{2}ikE_2 = \sqrt{2}M\Phi_0, \quad a_mB_2 - \sqrt{2}i(kB_3 - \epsilon E_1) = \sqrt{2}M\Phi_1, \\ -a_{m+1}B_1 - b_{m-1}B_3 + \sqrt{2}i\epsilon E_2 = \sqrt{2}M\Phi_2, \quad b_mB_2 + \sqrt{2}i(kB_1 + \epsilon E_3) = \sqrt{2}M\Phi_3; \\ a_m\Phi_0 - \sqrt{2}i\epsilon\Phi_1 = \sqrt{2}ME_1, \quad -\sqrt{2}i(k\Phi_0 + \epsilon\Phi_2) = \sqrt{2}ME_2, \\ -b_m\Phi_0 - \sqrt{2}i\epsilon\Phi_3 = \sqrt{2}ME_3, \quad -b_m\Phi_2 + \sqrt{2}ik\Phi_3 = \sqrt{2}MB_1, \\ b_{m-1}\Phi_1 + a_{m+1}\Phi_3 = \sqrt{2}MB_2, \quad -a_m\Phi_2 - \sqrt{2}ik\Phi_1 = \sqrt{2}MB_3. \end{gathered} \tag{14}$$

Projective operator method. In order to solve system (14) we will use the Fedorov – Gronskiy method based on projective operators [1]. We start with the operator of the third projection of the spin

$Y = -i\bar{J}^{12}$; it is readily verified that it satisfies the minimal equation $Y(Y-1)(Y+1) = 0$. This minimal equation permits us to introduce three projective operators:

$$P_0 = 1 - Y^2, \quad P_{+1} = \frac{1}{2}Y(Y+1), \quad P_{-1} = \frac{1}{2}Y(Y-1). \tag{15}$$

Accordingly, the complete wave function can be expanded into the sum of three parts:

$$\Psi = \Psi_0 + \Psi_{+1} + \Psi_{-1}, \quad \Psi_\sigma = P_\sigma \Psi, \quad \sigma = 0, +1, -1. \tag{16}$$

With the use of the explicit form of the generator $Y = -i\bar{J}^{12}$, we find the following structure for three projective constituents (these expressions refer to the cyclic basis)

$$\Psi_0 = \begin{vmatrix} \Phi_0(r) \\ 0 \\ \Phi_2(r) \\ 0 \\ 0 \\ E_2(r) \\ 0 \\ 0 \\ B_2(r) \\ 0 \end{vmatrix}, \quad \Psi_{+1} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ \Phi_3(r) \\ 0 \\ 0 \\ E_3(r) \\ B_1(r) \\ 0 \\ 0 \end{vmatrix}, \quad \Psi_{-1} = \begin{vmatrix} 0 \\ \Phi_1(r) \\ 0 \\ 0 \\ E_1(r) \\ 0 \\ 0 \\ 0 \\ 0 \\ B_3(r) \end{vmatrix}. \tag{17}$$

The above system of 10 equations (14) can be written in the matrix form. Let us act on this system by projective operators (15); in this way, we obtain three subsystems:

$$P_0, \quad \begin{array}{l} b_{m-1}E_1 - i\sqrt{2}E_2 k - a_{m+1}E_3 = \sqrt{2}M\Phi_0, \\ i\sqrt{2}\epsilon E_2 - a_{m+1}B_1 - b_{m-1}B_3 = \sqrt{2}M\Phi_2, \\ -i\sqrt{2}(k\Phi_0 + \epsilon\Phi_2) = \sqrt{2}ME_2, \quad b_{m-1}\Phi_1 + a_{m+1}\Phi_3 = \sqrt{2}MB_2; \end{array} \tag{18}$$

$$P_{+1}, \quad \begin{array}{l} i\sqrt{2}\epsilon E_3 + i\sqrt{2}kB_1 + b_m B_2 = \sqrt{2}M\Phi_3, \\ -b_m\Phi_0 - i\sqrt{2}\epsilon\Phi_3 = \sqrt{2}ME_3, \quad -b_m\Phi_2 + i\sqrt{2}k\Phi_3 = \sqrt{2}MB_1; \end{array} \tag{19}$$

$$P_{-1}, \quad \begin{array}{l} i\sqrt{2}\epsilon E_1 + a_m B_2 - i\sqrt{2}kB_3 = \sqrt{2}M\Phi_1, \\ a_m\Phi_0 - i\sqrt{2}\epsilon\Phi_1 = \sqrt{2}ME_1, \quad -i\sqrt{2}k\Phi_1 - a_m\Phi_2 = \sqrt{2}MB_3. \end{array} \tag{20}$$

In accordance with the Fedorov – Gronskiy method [1, 2], each projective component from (17) has to be determined by only one function:

$$\Psi_0(r) = \begin{vmatrix} \Phi_0 \\ 0 \\ \Phi_2 \\ 0 \\ 0 \\ E_2 \\ 0 \\ 0 \\ B_2 \\ 0 \end{vmatrix} f_0(r), \quad \Psi_{+1}(r) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ \Phi_3 \\ 0 \\ 0 \\ E_3 \\ B_1 \\ 0 \\ 0 \end{vmatrix} f_{+1}(r), \quad \Psi_{-1}(r) = \begin{vmatrix} 0 \\ \Phi_1 \\ 0 \\ 0 \\ E_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ B_3 \end{vmatrix} f_{-1}(r), \tag{21}$$

where the columns are composed of some numerical coefficients. So we reduce the systems to another form

$$P_0, \qquad \begin{aligned} &E_1 b_{m-1} f_{-1} - i\sqrt{2}kE_2 f_0 - E_3 a_{m+1} f_{+1} = \sqrt{2}M\Phi_0 f_0, \\ &i\sqrt{2}\epsilon E_2 f_0 - B_1 a_{m+1} f_{+1} - B_3 b_{m-1} f_{-1} = \sqrt{2}M\Phi_2 f_0, \\ &-i\sqrt{2}(k\Phi_0 f_0 + \epsilon\Phi_2 f_0) = \sqrt{2}ME_2 f_0, \quad \Phi_1 b_{m-1} f_{-1} + \Phi_3 a_{m+1} f_{+1} = \sqrt{2}MB_2 f_0; \end{aligned} \tag{22}$$

$$P_{+1}, \qquad \begin{aligned} &i\sqrt{2}\epsilon E_3 f_{+1} + i\sqrt{2}kB_1 f_{+1} + B_2 b_m f_0 = \sqrt{2}M\Phi_3 f_{+1}, \\ &-\Phi_0 b_m f_0 - i\sqrt{2}\epsilon\Phi_3 f_{+1} = \sqrt{2}ME_3 f_{+1}, \quad -\Phi_2 b_m f_0 + i\sqrt{2}k\Phi_3 f_{+1} = \sqrt{2}MB_1 f_{+1}; \end{aligned} \tag{23}$$

$$P_{-1}, \qquad \begin{aligned} &i\sqrt{2}\epsilon E_1 f_{-1} + B_2 a_m f_0 - i\sqrt{2}kB_3 f_{-1} = \sqrt{2}M\Phi_1 f_{-1}, \\ &\Phi_0 a_m f_0 - i\sqrt{2}\epsilon\Phi_1 f_{-1} = \sqrt{2}ME_1 f_{-1}, \quad -i\sqrt{2}k\Phi_1 f_{-1} - \Phi_2 a_m f_0 = \sqrt{2}MB_3 f_{-1}. \end{aligned} \tag{24}$$

Besides, according to the Fedorov – Gronsky method, equations (22)–(24) are to be consistent with the following differential constraints

$$b_{m-1} f_{-1}(r) = C_1 f_0(r), \quad a_m f_0(r) = C_2 f_{-1}(r), \quad a_{m+1} f_{+1}(r) = C_3 f_0(r), \quad b_m f_0(r) = C_4 f_{+1}(r), \tag{25}$$

where C_1, C_2, C_3, C_4 are some numerical parameters. Relations (25) will allow us to transform the differential equations (22)–(24) to the algebraic form. In this way, we get

$$P_0, \qquad \begin{aligned} &E_1 C_1 - i\sqrt{2}kE_2 - E_3 C_3 = \sqrt{2}M\Phi_0, \\ &i\sqrt{2}\epsilon E_2 - B_1 C_3 - B_3 C_1 = \sqrt{2}M\Phi_2, \\ &-i\sqrt{2}(k\Phi_0 + \epsilon\Phi_2) = \sqrt{2}ME_2, \quad \Phi_1 C_1 + \Phi_3 C_3 = \sqrt{2}MB_2; \end{aligned} \tag{26}$$

$$P_{+1}, \qquad \begin{aligned} &i\sqrt{2}\epsilon E_3 + i\sqrt{2}kB_1 + B_2 C_4 = \sqrt{2}M\Phi_3, \\ &-\Phi_0 C_4 - i\sqrt{2}\epsilon\Phi_3 = \sqrt{2}ME_3, \quad -\Phi_2 C_4 + i\sqrt{2}k\Phi_3 = \sqrt{2}MB_1; \end{aligned} \tag{27}$$

$$P_{-1}, \qquad \begin{aligned} &i\sqrt{2}\epsilon E_1 + B_2 C_2 - i\sqrt{2}kB_3 = \sqrt{2}M\Phi_1, \\ &\Phi_0 C_2 - i\sqrt{2}\epsilon\Phi_1 = \sqrt{2}ME_1, \quad -i\sqrt{2}k\Phi_1 - \Phi_2 C_2 = \sqrt{2}MB_3. \end{aligned} \tag{28}$$

Note that in the massless case, these equations take a simpler form:

$$P_0, \qquad \begin{aligned} &E_1 C_1 - i\sqrt{2}kE_2 - E_3 C_3 = 0, \\ &i\sqrt{2}\epsilon E_2 - B_1 C_3 - B_3 C_1 = 0, \\ &-i\sqrt{2}(k\Phi_0 + \epsilon\Phi_2) = \sqrt{2}E_2, \quad \Phi_1 C_1 + \Phi_3 C_3 = \sqrt{2}B_2; \end{aligned} \tag{29}$$

$$P_{+1}, \qquad \begin{aligned} &i\sqrt{2}\epsilon E_3 + i\sqrt{2}kB_1 + B_2 C_4 = 0, \\ &-\Phi_0 C_4 - i\sqrt{2}\epsilon\Phi_3 = \sqrt{2}E_3, \quad -\Phi_2 C_4 + i\sqrt{2}k\Phi_3 = \sqrt{2}B_1; \end{aligned} \tag{30}$$

$$P_{-1}, \qquad \begin{aligned} &i\sqrt{2}\epsilon E_1 + B_2 C_2 - i\sqrt{2}kB_3 = 0, \\ &\Phi_0 C_2 - i\sqrt{2}\epsilon\Phi_1 = \sqrt{2}E_1, \quad -i\sqrt{2}k\Phi_1 - \Phi_2 C_2 = \sqrt{2}B_3; \end{aligned} \tag{31}$$

we will turn to considering the massless case later. From constraints (25) one can derive the second order equations:

$$b_{m-1}a_m f_0(r) = C_1 C_2 f_0(r), \quad a_m b_{m-1} f_{-1}(r) = C_2 C_1 f_{-1}(r),$$
$$a_{m+1}b_m f_0(r) = C_3 C_4 f_0(r), \quad b_m a_{m+1} f_{+1}(r) = C_4 C_3 f_{+1}(r). \tag{32}$$

Taking in mind the linearity of eqs. (25), we conclude that the parameters in each pair can be chosen as equal: $C_2 = C_1$, $C_4 = C_3$. Therefore the constraints and the second order equations take on the form

$$b_{m-1} f_{-1}(r) = C_1 f_0(r), \quad a_m f_0(r) = C_1 f_{-1}(r),$$
$$a_{m+1} f_{+1}(r) = C_3 f_0(r), \quad b_m f_0(r) = C_3 f_{+1}(r); \tag{33}$$

$$\left(b_{m-1}a_m - C_1^2\right) f_0 = 0, \quad \left(a_m b_{m-1} - C_1^2\right) f_{-1} = 0,$$
$$\left(a_{m+1}b_m - C_3^2\right) f_0 = 0, \quad \left(b_m a_{m+1} - C_3^2\right) f_{+1} = 0. \tag{34}$$

Further we get (for brevity we use the notation $eB \Rightarrow B$)

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2 r^2}{4} - \frac{m^2}{r^2} - Bm + B - C_1^2\right) f_0 = 0,$$
$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2 r^2}{4} - \frac{m^2}{r^2} - Bm - B - C_3^2\right) f_0 = 0, \quad \Rightarrow \quad C_3^2 = C_1^2 - 2B, \tag{35}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2 r^2}{4} - \frac{(m-1)^2}{r^2} - Bm - C_1^2\right) f_{-1} = 0,$$
$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2 r^2}{4} - \frac{(m+1)^2}{r^2} - Bm - C_3^2\right) f_{+1} = 0.$$

Thus, we have only three different equations and the identity $C_3^2 = C_1^2 - 2B$. Let us introduce the new parameter $X = B - C_1^2$, then the three equations are written as follows (we transform them into the variable $x = Br^2/2$)

$$1, \quad \left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{4} - \frac{(m/2)^2}{x^2} + \frac{1}{x}\left(-\frac{m}{2} + \frac{X}{2B}\right)\right) f_0 = 0,$$
$$2, \quad \left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{4} - \frac{[(m-1)/2]^2}{x^2} + \frac{1}{x}\left(-\frac{m+1}{2} + \frac{X}{2B}\right)\right) f_{-1} = 0, \tag{36}$$
$$3, \quad \left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{4} - \frac{[(m+1)/2]^2}{x^2} + \frac{1}{x}\left(-\frac{m-1}{2} + \frac{X}{2B}\right)\right) f_{+1} = 0.$$

These three equations are of the same type. It is enough to investigate the first, and the results for another two can be obtained by formal changes. Let us search solutions of the first equation in the form $f_0(x) = x^A e^{Cx} F(x)$; the equation for $F(x)$ reads

$$xF'' + (2A + 1 + 2Cx)F' + \left\{\frac{1}{x}\left(A^2 - (m/2)^2\right) + \left(2AC + C - \frac{m}{2} + \frac{X}{2B}\right) + x\left(C^2 - \frac{1}{4}\right)\right\} F = 0;$$

impose the restrictions

$$A^2 - (m/2)^2 = 0 \Rightarrow A = \pm|m/2|, \quad C^2 - \frac{1}{4} = 0 \Rightarrow C = \pm\frac{1}{2}.$$

To get solutions that vanish at the point $r = 0$ and in infinity, we choose $A = +|m/2|$, $C = -1/2$; so the equation simplifies

$$xF'' + (|m|+1-x)F' - \left(\frac{|m|+m}{2} + \frac{1}{2} - \frac{X}{2B}\right)F = 0.$$

This is an equation of the confluent hypergeometric type with parameters

$$a = \frac{|m|+m}{2} + \frac{1}{2} - \frac{X}{2B}, \quad c = |m|+1, \quad F = \Phi(a,c,x). \tag{37}$$

The polynomial condition $a = -n_1$ gives

$$1, \qquad X = +2B\left(\frac{|m|+m}{2} + \frac{1}{2} + n_1\right) > 0, \quad n_1 = 0,1,2,\ldots; \tag{38}$$

to this spectrum there correspond the following solutions

$$1, \quad f_0(x) = x^{+\frac{|m|}{2}} e^{-x/2} F_1(x), \quad F_1(x) = \Phi(-n_1, |m|+1, x). \tag{39}$$

The other two equations lead to similar results. So we have

$$1, \quad \begin{array}{l} f_0(x) = x^{+\frac{|m|}{2}} e^{-x/2} F_1(x), \quad F_1(x) = \Phi(-n_1, |m|+1, x), \\ X = 2B\left(\frac{|m|+m}{2} + \frac{1}{2} + n_1\right) > B, \quad n_1 = 0,1,2,\ldots; \end{array} \tag{40}$$

$$2, \quad \begin{array}{l} f_{-1}(x) = x^{+\frac{|m-1|}{2}} e^{-x/2} F_2(x), \quad F_2(x) = \Phi(-n_2, |m-1|+1, x), \\ X = 2B\left(\frac{|m-1|+m+1}{2} + \frac{1}{2} + n_2\right) > B, \quad n_2 = 0,1,2,\ldots; \end{array} \tag{41}$$

$$3, \quad \begin{array}{l} f_{+1}(x) = x^{+\frac{|m+1|}{2}} e^{-x/2} F_3(x), \quad F_3(x) = \Phi(-n_3, |m+1|+1, x), \\ X = 2B\left(\frac{|m+1|+m-1}{2} + \frac{1}{2} + n_3\right) > B, \quad n_3 = 0,1,2,\ldots. \end{array} \tag{42}$$

Note that the X value in all three cases (40)–(42) should be the same. Therefore, we can use the simplest quantization rule for the quantity X:

$$X = 2BN > 0, \quad N = \frac{|m|+m}{2} + \frac{1}{2} + n, \quad n = 0,1,2,\ldots. \tag{43}$$

Write down the expressions for C_1, C_3:

$$C_1 = \delta_1 i\sqrt{X-B}, \quad C_3 = \delta_3 i\sqrt{X+B}, \quad \delta_1^2 = 1, \quad \delta_3^2 = 1; \tag{44}$$

the quantities C_1 and C_3 turn out to be purely imaginary; the quantities δ_1 and δ_3 are still considered as independent. However, it can be proven that different choices of signs do not matter because they lead to equivalent results which differ only by simple linear transformations.

The study of the algebraic system. Let us turn to the algebraic system in the presence of a uniform magnetic field. We will follow the case $\delta_1 = +1$, $\delta_3 = +1$. We write the system in the matrix form $AZ = 0$; from vanishing the determinant of the matrix, we obtain

$$\det A = 32M^2(k^2 + M^2 + X - \epsilon^2)\left(B^2(k-\epsilon)(k+\epsilon) + M^2(k^2 + M^2 + X - \epsilon^2)^2\right) = 0; \tag{45}$$

the variable X can take three different values. In dimensionless variables

$$\frac{\epsilon}{M}=E, \quad \frac{k}{M}=K, \quad \frac{B}{M^2}=b, \quad \frac{X_i}{M^2}=x_i, \quad i=1,2,3, \tag{46}$$

these roots are given by the formulas

$$x_1=E^2-K^2-1, \quad x_2=-b\sqrt{E^2-K^2}+E^2-K^2-1, \quad x_3=+b\sqrt{E^2-K^2}+E^2-K^2-1. \tag{47}$$

Eqs. (47) can be solved with respect to the energy parameter (taking in mind that $E^2 - K^2 > 0$):

$$E^2-K^2=1+2bN, \quad E^2-K^2=\frac{1}{4}\left(b\pm\sqrt{b^2+8bN+4}\right)^2. \tag{48}$$

Let us consider this system at $x = x_1$. We verify that the rank of the matrix is equal to 9. If we remove the first row, then the rank of the remaining matrix will be the same. We verify that by removing the first column, we get a 9 × 9 matrix with a non-vanishing determinant. Thus, we obtain the inhomogeneous system, its solutions have the form

$$x_1, \quad \begin{vmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\\ E_1\\ E_2\\ E_3\\ B_1\\ B_2\\ B_3\end{vmatrix}=\Phi_0\begin{vmatrix}0\\ \frac{E}{K}\\ 0\\ -i\frac{\sqrt{E^2-K^2-1-b}}{\sqrt{2}}\\ -i\frac{E^2-K^2}{K}\\ i\frac{\sqrt{E^2-K^2-1+b}}{\sqrt{2}}\\ -i\frac{E\sqrt{E^2-K^2-1+b}}{\sqrt{2}K}\\ 0\\ -i\frac{E\sqrt{E^2-K^2-1-b}}{\sqrt{2}K}\end{vmatrix}. \tag{49}$$

Solutions referring to the roots x_2 and x_3 can be found similarly. In relation (49) the quantity Φ_0 can be arbitrary, for example, $\Phi_0 = 1$. So, we have found three series of energy and respective exact wave functions.

Massless case. In the massless case (see (29)–(31)), interaction with the magnetic field should be excluded; so the second order equations are written as follows (note that $X = -C_1^2$):

$$\begin{aligned}
&1, \quad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+X\right)f_0=0,\\
&2, \quad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(m-1)^2}{r^2}+X\right)f_{-1}=0,\\
&3, \quad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(m+1)^2}{r^2}+X\right)f_{+1}=0.
\end{aligned} \tag{50}$$

In order to have solutions of eqs. (50) consistent with the gauge solutions (which will be specified below), the parameter X should be equal to $-C_1^2 = X = \epsilon^2 - k^2$. In the variable $z = \sqrt{X}r$, three equations from (50) take the Bessel form

$$\begin{aligned}
&1, \quad \left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1-\frac{m^2}{z^2}\right)f_0(z)=0, \quad f_0(z)=L_1 J_{\pm|m|}(z);\\
&2, \quad \left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1-\frac{(m-1)^2}{z^2}\right)f_{-1}(z)=0, \quad f_{-1}(z)=L_2 J_{\pm|m-1|}(z);\\
&3, \quad \left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1-\frac{(m+1)^2}{z^2}\right)f_{+1}(z)=0, \quad f_{+1}(z)=L_3 J_{\pm|m+1|}(z).
\end{aligned} \tag{51}$$

Further, we will consider solutions that are regular at the point $z = 0$:

$$f_0(z)=L_1 J_{|m|}(z), \quad f_{-1}(z)=L_2 J_{|m-1|}(z), \quad f_{+1}(z)=L_3 J_{|m+1|}(z). \tag{52}$$

Let us transform the above constraints (taking in mind $C_1 = C_2 = C_3 = \sqrt{-X}$; now the operators of the first order do not contain the term $Br^2/2$)

$$\begin{aligned}
b_{m-1} f_{-1}(r) = \sqrt{-X} f_0(r), \quad a_m f_0(r) = \sqrt{-X} f_{-1}(r),\\
a_{m+1} f_{+1}(r) = \sqrt{-X} f_0(r), \quad b_m f_0(r) = \sqrt{-X} f_{+1}(r)
\end{aligned}$$

to the variable z, then we get the following equations

$$\begin{aligned}
&\left(\frac{d}{dz}-\frac{m-1}{z}\right)L_2 J_{|m-1|}(z) = iL_1 J_{|m|}(z), \quad \left(\frac{d}{dz}+\frac{m}{z}\right)L_1 J_{|m|}(z) = iL_2 J_{|m-1|}(z),\\
&\left(\frac{d}{dz}+\frac{m+1}{z}\right)L_3 J_{|m+1|}(z) = iL_1 J_{|m|}(z), \quad \left(\frac{d}{dz}-\frac{m}{z}\right)L_1 J_{|m|}(z) = iL_3 J_{|m+1|}(z).
\end{aligned} \tag{53}$$

With the use of the well-known formulas for the Bessel functions

$$\left(\frac{d}{dz}+\frac{p}{z}\right)J_p = J_{p-1}, \quad \left(\frac{d}{dz}-\frac{p}{z}\right)J_p = -J_{p+1},$$

we derive the linear relations

$$\begin{aligned}
&A \qquad f_0(z)=L_1 J_m(z), \quad f_{-1}(z)=-iL_1 J_{m-1}(z), \quad f_{+1}(z)=+iL_1 J_{m+1}(z);\\
&B \qquad f_0(z)=L_1' J_{-m}(z), \quad f_{-1}(z)=+iL_1' J_{-m+1}(z), \quad f_{+1}(z)=-iL_1' J_{-m-1}(z).
\end{aligned} \tag{54}$$

Let us turn to the algebraic system. In the massless case, the system (29)–(31) takes the form

$$C_1 = C_2 = C_3 = C_4 = \sqrt{-X} = \mu = i\sqrt{\epsilon^2 - k^2};$$

$$\begin{aligned}
&P_0, \quad \begin{aligned}&\mu E_1 - i\sqrt{2}kE_2 - \mu E_3 = 0, \quad i\sqrt{2}\epsilon E_2 - \mu B_1 - \mu B_3 = 0,\\ &-i\sqrt{2}(k\Phi_0 + \epsilon\Phi_2) = \sqrt{2}E_2, \quad \mu\Phi_1 + \mu\Phi_3 = \sqrt{2}B_2;\end{aligned}\\
&P_{+1}, \quad \begin{aligned}&i\sqrt{2}\epsilon E_3 + i\sqrt{2}kB_1 + \mu B_2 = 0,\\ &-\mu\Phi_0 - i\sqrt{2}\epsilon\Phi_3 = \sqrt{2}E_3, \quad -\mu\Phi_2 + i\sqrt{2}k\Phi_3 = \sqrt{2}B_1;\end{aligned}\\
&P_{-1}, \quad \begin{aligned}&i\sqrt{2}\epsilon E_1 + \mu B_2 - i\sqrt{2}kB_3 = 0,\\ &\mu\Phi_0 - i\sqrt{2}\epsilon\Phi_1 = \sqrt{2}E_1, \quad -i\sqrt{2}k\Phi_1 - \mu\Phi_2 = \sqrt{2}B_3.\end{aligned}
\end{aligned} \tag{55}$$

This system is written in the matrix form $\Psi = \{\Phi_0, \Phi_1, \Phi_2, \Phi_3; E_1, E_2, E_3, B_1, B_2, B_3\}$, $A_{10\times 10}\Psi = 0$; The rank of the matrix A equals to 7; we may verify that when removing the rows 2, 3, 4 the rank of the matrix remains the same. Therefore, we have the following system

$$\begin{vmatrix} 0 & 0 & 0 & 0 & i\sqrt{\epsilon^2-k^2} & -i\sqrt{2}k & -i\sqrt{\epsilon^2-k^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2}\epsilon & i\sqrt{2}k & i\sqrt{\epsilon^2-k^2} & 0 \\ -i\sqrt{\epsilon^2-k^2} & 0 & 0 & -i\sqrt{2}\epsilon & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{\epsilon^2-k^2} & i\sqrt{2}k & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{2}\epsilon & 0 & 0 & 0 & i\sqrt{\epsilon^2-k^2} & -i\sqrt{2}k \\ i\sqrt{\epsilon^2-k^2} & -i\sqrt{2}\epsilon & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -i\sqrt{2}k & -i\sqrt{\epsilon^2-k^2} & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \end{vmatrix} \begin{vmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3. \end{vmatrix} = 0.$$

Note that if in the matrix we remove the column 2, 3, 4, then we will get the 7 × 7 matrix with the non-vanishing determinant, $\det A_{7\times 7} = 8\sqrt{2}k\epsilon(k^2-\epsilon^2)$; thus, we obtain the inhomogeneous system with the following general solution

$$\begin{gathered} \Phi_0 = \frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_1 - \frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_3 - \frac{k}{\epsilon}\Phi_2, \\ E_1 = -\frac{i(k^2+\epsilon^2)}{2\epsilon}\Phi_1 + \frac{i(k^2-\epsilon^2)}{2\epsilon}\Phi_3 - \frac{ik\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_2, \\ E_2 = -\frac{ik\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_1 + \frac{i(k^2-\epsilon^2)}{\epsilon}\Phi_2 + \frac{ik\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_3, \\ E_3 = \frac{i(k^2-\epsilon^2)}{2\epsilon}\Phi_1 + \frac{ik\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_2 - \frac{i(k^2+\epsilon^2)}{2\epsilon}\Phi_3, \\ B_1 = ik\Phi_3 - \frac{i\sqrt{\epsilon^2-k^2}}{\sqrt{2}}\Phi_2, \quad B_2 = \frac{i\sqrt{\epsilon^2-k^2}}{\sqrt{2}}\Phi_1 + \frac{i\sqrt{\epsilon^2-k^2}}{\sqrt{2}}\Phi_3, \\ B_3 = -\frac{i\sqrt{\epsilon^2-k^2}}{\sqrt{2}}\Phi_2 - ik\Phi_1. \end{gathered} \tag{56}$$

Another method of solving the system is possible. Indeed, eliminating the tensor components

$$\begin{gathered} E_2 = -i(k\Phi_0 + \epsilon\Phi_2), \quad B_2 = \frac{\mu}{\sqrt{2}}(\Phi_1 + \Phi_3), \\ E_3 = -\frac{\mu}{\sqrt{2}}\Phi_0 - i\epsilon\Phi_3, \quad B_1 = -\frac{\mu}{\sqrt{2}}\Phi_2 + ik\Phi_3, \\ E_1 = \frac{\mu}{\sqrt{2}}\Phi_0 - i\epsilon\Phi_1, \quad B_3 = -ik\Phi_1 - \frac{\mu}{\sqrt{2}}\Phi_2 \end{gathered} \tag{57}$$

from four remaining equations

$$\mu E_1 - i\sqrt{2}kE_2 - \mu E_3 = 0, \quad i\sqrt{2}\epsilon E_2 - \mu B_1 - \mu B_3 = 0,$$

$$i\sqrt{2}\epsilon E_3 + i\sqrt{2}kB_1 + \mu B_2 = 0, \quad i\sqrt{2}\epsilon E_1 + \mu B_2 - i\sqrt{2}kB_3 = 0,$$

we get the homogenous system for 4 variables (taking in mind $\mu = i\sqrt{\epsilon^2-k^2}$)

$$\begin{vmatrix} -\sqrt{2}\epsilon^2 & \epsilon\sqrt{\epsilon^2-k^2} & -\sqrt{2}k\epsilon & -\epsilon\sqrt{\epsilon^2-k^2} \\ \sqrt{2}k\epsilon & -k\sqrt{\epsilon^2-k^2} & \sqrt{2}k^2 & k\sqrt{\epsilon^2-k^2} \\ \epsilon\sqrt{\epsilon^2-k^2} & \frac{k^2-\epsilon^2}{\sqrt{2}} & k\sqrt{\epsilon^2-k^2} & \frac{\epsilon^2-k^2}{\sqrt{2}} \\ -\epsilon\sqrt{\epsilon^2-k^2} & \frac{\epsilon^2-k^2}{\sqrt{2}} & -k\sqrt{\epsilon^2-k^2} & \frac{k^2-\epsilon^2}{\sqrt{2}} \end{vmatrix} \begin{vmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{vmatrix} = 0.$$

We verify that the rank of the matrix equals to 1. Without changing the rank of the matrix, we can remove the rows 2, 3, 4. Thus, we get the expression for Φ_0:

$$\Phi_0 = \frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_1 - \frac{k}{\epsilon}\Phi_2 - \frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_3. \tag{58}$$

If we substitute expression (58) in the formula (57), for the tensor components we find

$$\begin{aligned} &E_2 = -i\frac{k\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_1 - i\frac{\epsilon^2-k^2}{\epsilon}\Phi_2 + i\frac{k\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon}\Phi_3, \quad B_2 = i\frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}}\Phi_1 + i\frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}}\Phi_3; \\ &E_3 = -i\frac{\epsilon^2-k^2}{2\epsilon}\Phi_1 + i\frac{\sqrt{\epsilon^2-k^2}k}{\sqrt{2}\epsilon}\Phi_2 - i\frac{k^2+\epsilon^2}{2\epsilon}\Phi_3, \quad B_1 = -i\frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}}\Phi_2 + ik\Phi_3; \\ &E_1 = -i\frac{\epsilon^2+k^2}{2\epsilon}\Phi_1 - i\frac{\sqrt{\epsilon^2-k^2}k}{\sqrt{2}\epsilon}\Phi_2 - i\frac{\epsilon^2-k^2}{2\epsilon}\Phi_3, \quad B_3 = -ik\Phi_1 - i\frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}}\Phi_2. \end{aligned} \tag{59}$$

Let us examine three solutions for tensor components, referring to the variants $(\Phi_1 = 1,0,0);\ (0,\Phi_2 = 1,0);\ (0,0,\Phi_3 = 1)$; making up the 3 × 6 matrix

$$\begin{vmatrix} -i\frac{\epsilon^2+k^2}{2\epsilon} & -i\frac{k\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon} & -i\frac{\epsilon^2-k^2}{2\epsilon} & 0 & i\frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}} & -ik \\ -i\frac{\sqrt{\epsilon^2-k^2}k}{\sqrt{2}\epsilon} & -i\frac{\epsilon^2-k^2}{\epsilon} & +i\frac{\sqrt{\epsilon^2-k^2}k}{\sqrt{2}\epsilon} & -i\frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}} & 0 & -i\frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}} \\ -i\frac{\epsilon^2-k^2}{2\epsilon} & +i\frac{k\sqrt{\epsilon^2-k^2}}{\sqrt{2}\epsilon} & -i\frac{k^2+\epsilon^2}{2\epsilon} & +ik & i\frac{\sqrt{\epsilon^2-k^2}}{\sqrt{2}} & 0 \end{vmatrix}.$$

We can see that rows in this matrix are linearly dependent:

$$(Row\ 1) + \frac{-\sqrt{2}k}{\sqrt{\epsilon^2-k^2}} \times (Row\ 2) = Row\ 3; \tag{60}$$

this means that among three independent solutions (58) one has the gauge nature.

The possibility exists to find another gauge solution as a 4-gradient of the scalar function. Let the scalar function $\Phi(x)$ obey the massless Klein – Fock – Gordon wave equation in cylindric coordinates, then explicitly it reads

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} - \frac{\partial^2}{\partial z^2}\right)e^{-i\epsilon t}e^{im\phi}e^{ikz}f(r) = 0, \tag{61}$$

whence it follows an equation of the Bessel form (we use the variable $x = \sqrt{\epsilon^2-k^2}r$)

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1-\frac{m^2}{x^2}\right)f=0,\quad f=LJ_{\pm|m|}(x).$$

Tetrad components of this gauge solution are determined by the formula $\Phi_l(x)=e^{\alpha}_{(l)}\partial_{\alpha}\Phi(x)$, or in detail

$$\Phi_0=\partial_t\Phi(x),\quad \Phi_1=\partial_r\Phi(x),\quad \Phi_2=\frac{1}{r}\partial_{\phi}\Phi(x),\quad \Phi_3=\partial_z\Phi(x).$$

Taking in mind the substitution for Φ, we get

$$\Phi_0=-i\epsilon f,\quad \Phi_1=\frac{d}{dr}f,\quad \Phi_2=\frac{im}{r}f,\quad \Phi_3=ikf. \tag{62}$$

This 4-vector should be transformed to the cyclic basis

$$\bar\Phi_0=\Phi_0,\quad \bar\Phi_2=\Phi_3,\quad \bar\Phi_1=-\frac{1}{\sqrt2}\Phi_1+\frac{i}{\sqrt2}\Phi_2,\quad \bar\Phi_3=\frac{1}{\sqrt2}\Phi_1+\frac{i}{\sqrt2}\Phi_2. \tag{63}$$

Whence we derive

$$\bar\Phi_0=-i\epsilon J_{|m|},\quad \bar\Phi_2=ikJ_{|m|},\quad \bar\Phi_1=-\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}\left(\frac{d}{dz}+\frac{m}{z}\right)J_{|m|},\quad \bar\Phi_3=\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}\left(\frac{d}{dz}-\frac{m}{z}\right)J_{|m|}. \tag{64}$$

It is convenient to examine separately two possibilities:

$$\begin{aligned}
&I,\quad \bar\Phi_0=-i\epsilon J_m,\quad \bar\Phi_2=ikJ_m,\quad \bar\Phi_1=-\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}\left(\frac{d}{dz}+\frac{m}{z}\right)J_m,\quad \bar\Phi_3=\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}\left(\frac{d}{dz}-\frac{m}{z}\right)J_m;\\
&II,\quad \bar\Phi_0=-i\epsilon J_{-m},\quad \bar\Phi_2=ikJ_{-m},\quad \bar\Phi_1=-\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}\left(\frac{d}{dz}+\frac{m}{r}\right)J_{-m},\quad \bar\Phi_3=\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}\left(\frac{d}{dz}-\frac{m}{z}\right)J_{-m}.
\end{aligned} \tag{65}$$

Whence, with the use of the known properties (58) of Bessel functions, we find expressions for relative coefficients:

$$\begin{aligned}
&I,\quad \bar\Phi_0=-i\epsilon LJ_m,\quad \bar\Phi_2=ikLJ_m,\quad \bar\Phi_1=-\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}LJ_{m-1},\quad \bar\Phi_3=-\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}LJ_{m+1};\\
&II,\quad \bar\Phi_0=-i\epsilon LJ_{-m},\quad \bar\Phi_2=ikLJ_{-m},\quad \bar\Phi_1=\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}LJ_{-m+1},\quad \bar\Phi_3=\frac{\sqrt{\epsilon^2-k^2}}{\sqrt2}LJ_{-m-1};
\end{aligned} \tag{66}$$

where we may set $L=1$ and $L'=1$. Thus, we have found four independent solutions in the massless case, two of them are gauge ones.

Conclusions. The system of equations describing a spin 1 particle was studied in cylindric coordinates with the use of tetrad formalism and matrix 10-dimension Duffin – Kemmer – Petieau formalism. After separating the variables, to resolve the system of 10 equations in the variable r, we applied the the Fedorov – Gronskiy method based on the use of projective operators. In the presence of an external uniform magnetic field, we constructed in an explicit form three independent classes of wave functions with the corresponding energy spectra. Separately the massless field with spin 1 was studied; there were found four linearly independent solutions, two of which are gauge ones, and other two do not contain gauge degrees of freedom.

It should be noted that all four solutions for a massless spin 1 particle are necessary when constructing the gauge solutions for a massless spin 2 particle in accordance with the Pauli – Fierz approach [6, 7].

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