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FOURIER SERIES FOR THE MULTIDIMENSIONAL-MATRIX FUNCTIONS OF THE VECTOR VARIABLE

Abstract. In the article, the theory of the Fourier series on the orthogonal multidimensional-matrix (mdm) polynomials is developed. The known results from the theory of the orthogonal polynomials of the vector variable and the Fourier series are given and the new results are presented. In particular, the known results of the Fourier series theory are extended to the case of the mdm functions, what allows us to solve more general approximation problems. The general case of the approximation of the mdm function of the vector argument by the Fourier series on the orthogonal mdm polynomials is realized programmatically as the program function and its efficiency is confirmed. The analytical expressions for the coefficients of the second degree orthogonal polynomials and Fourier series for possible analytical studies are obtained.

Keywords: Fourier series, multidimensional-matrix orthogonal polynomials, multivariate polynomial regression

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РЯДЫ ФУРЬЕ ДЛЯ МНОГОМЕРНО-МАТРИЧНЫХ ФУНКЦИЙ ВЕКТОРНОЙ ПЕРЕМЕННОЙ

Аннотация. В статье развивается теория рядов Фурье по ортогональным многомерно-матричным полиномам. Приводятся известные результаты теории ортогональных полиномов векторной переменной и рядов Фурье и представлены новые результаты. В частности, известные результаты теории рядов Фурье распространяются на случай многомерно-матричных функций, что позволяет решать более общие задачи аппроксимации. Выполнена программная реализация общего случая аппроксимации многомерно-матричной функции векторного аргумента рядом Фурье по ортогональным многомерно-матричным полиномам и подтверждена ее работоспособность. Получены также аналитические выражения коэффициентов полиномов и рядов Фурье второй степени для возможных аналитических исследований.

Ключевые слова: ряды Фурье, многомерно-матричные ортогональные полиномы, многомерная полиномиальная регрессия

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Introduction. The most important tool for research of the real systems and processes is approximation. The mathematical models of the real systems and processes are their approximate mathematical images. The various methods of approximation exist, one of which is approximation by Fourier series. This article is devoted to approximation of the functions of several variables by Fourier series on the orthogonal polynomials.

The history of the orthogonal polynomials of both one and several arguments dates back to Hermite [1]. Hermite in [1] and then P. Appel and Kampe de Feriet in [2] studied in details the properties of the so-called Hermite polynomials of one and two arguments. The general theory of the orthogonal polynomials of many arguments is developed in the work [3]. As per the works [1–3], this theory is constructed as the theory of two bi-orthonormal sequences of the polynomials: basic and conjugate. In the work [4], it is proposed to choose a polynomial with the unit coefficient at the highest degree as the basic polynomial. This theory uses the classical (scalar) mathematical approach and is therefore the classical theory. The classical theory can be found also in [5, 6]. The classical approach is characterized by bulkiness and poor formalization, which entail the practical loss of the theoretical and algorithmic generality.

In the current article, the multidimensional-matrix (mdm) approach is used that is free from these pointed above disadvantages. The foundations of the theory of the multidimensional matrices were laid by N. P. Sokolov in the works [7, 8]. This theory was developed later in the works [9, 10]. Sokolov's theory has a number of applications today, an overview of which can be found in the work [11]. These are vector multiconnected Markov chains, parallel factor analysis, multiindex linear programming problems, educational timetable problem, mdm statistical decisions, finite-dimensional moments of the stationary random sequences and their estimations, mdm principal components method, mdm data model in databases and OLAP-systems, mdm regression analysis and other applications.

The author of the current article is aware of the works related to the multidimensional matrices and appeared in recent time. Primarily, these are the works using the terms "multiway data" or "multiway arrays" (see, for example, [12, 13]). The authors of these works do not refer to the works of N. P. Sokolov It is not clear whether they are aware of the Sokolov's works or not. These works does not use the theory of N. P. Sokolov and have no own mathematical theory. Graphical (geometrical) representations of the multiway arrays prevail in these works. Secondly, it is the work [14] the author of which announces his own "multidimensional matrix mathematics". This approach also assumes the continuation of the development to the mathematical completion.

The current article is devoted to use the mdm theory of the orthogonal polynomials and the Fourier series of the vector argument [15, 10] in regression analysis. This theory is the combining of the ideas of the works [3, 4] on the basis of the mdm mathematical approach. It is how the mdm theory of the orthogonal polynomials and the Fourier series of the vector argument arose.

The basic definitions of the theory of the multidimensional matrices in English can be found in the appendix A3 to the current article and in the appendix to the article [16].

1. Orthogonal polynomials of the vector variable. Let Ω be some closed region of the space R^n, $\rho(x)$, $x \in \Omega$, be nonnegative function (weight function) such that the integrals (the moments of the weight function $\rho(x)$)

$$\nu_{x^i} = \int_{\Omega} x^i \rho(x) dx < \infty, \quad i = 0,1,2,..., \tag{1}$$

exist, and $L_2(\rho,\Omega)$ be the space of the functions with integrable square in Ω with the weight $\rho(x)$. Here x^i is the (0,0)-rolled degree of the one-dimensional matrix x: $x^i = {}^{0,0}(x^i) = {}^{0,0}(x \cdot x \cdots x)$ [A3, 16]. We will often avoid the notation (0,0)-rolled degree and will write x^i instead of ${}^{0,0}(x^i)$.

A mdm r degree polynomial $Q_r(x)$ of the vector (one-dimensional) variable $x \in \Omega$ is defined as follows [15, 10]:

$$Q_r(x) = \sum_{k=0}^{r} {}^{0,k}\left(C^*_{(r,k)} x^k\right) = \sum_{k=0}^{r} {}^{0,k}\left(x^k C^*_{(k,r)}\right), \quad r = 0,1,2,..., \tag{2}$$

where $C^*_{(r,k)}$ are the $(r + k)$-dimensional matrices of the coefficients,

$$C^*_{(r,k)} = \left(c^*_{i_1,...,i_r,j_1,...,j_k}\right), \quad r = 0,1,2,..., \quad k = 0,1,2,...,r,$$

symmetrical with respect to the indices of their two multiindices $(i_1,...,i_r)$, $(j_1,..., j_k)$ and satisfying the conditions

$$C^*_{(r,k)} = \left(C^*_{(k,r)}\right)^{H_{r+k,k}}, \quad C^*_{(k,r)} = \left(C^*_{(r,k)}\right)^{B_{k+r,k}}.$$

The notations $H_{r+k,k}$ and $B_{r+k,k}$ mean the transpose substitutions of the types "back" and "forward" respectively [10, A3, 16]. Each of the indices of the multiindices $(i_1,...,i_r)$, $(j_1,..., j_k)$ takes the values $1,2,...,n$.

D e f i n i t i o n. The sequence of the mdm polynomials $Q_r(x)$ (2) is called orthogonal in $L_2(\rho,\Omega)$ if the following conditions are satisfied:

$$\int_{\Omega} Q_r(x)Q_k(x)\rho(x)dx \begin{cases} = 0, & k = 0,1,...,r-1, \\ \neq 0, & k = r, \end{cases} \tag{3}$$

where $Q_r(x)Q_k(x)$ is the (0,0)-rolled product of the matrices $Q_r(x)$ and $Q_k(x)$: $Q_r(x)Q_k(x) = {}^{0,0}\left(Q_r(x)Q_k(x)\right)$. We will often avoid the notation (0,0)-rolled product and will write AB instead of ${}^{0,0}(AB)$.

The two sequences of the orthogonal polynomials of many variables are considered: the basic sequence $P_r(x)$ and the sequence $Q_r(x)$ conjugate of $P_r(x)$, $r = 0,1,2,...$.

D e f i n i t i o n. The mdm r degree polynomial in $L_2(\rho,\Omega)$ of the following form

$$P_r(x) = \sum_{k=0}^{r-1} {}^{0,k}\left(C_{(r,k)}x^k\right) + x^r = \sum_{k=0}^{r-1} {}^{0,k}\left(x^k C_{(k,r)}\right) + x^r, \quad r = 0,1,2,..., \tag{4}$$

is called the basic polynomial, where $C_{(r,k)}$ are $(r + k)$-dimensional matrices of the coefficients,

$$C_{(r,k)} = (c_{i_1,...,i_r,j_1,...,j_k}), \quad r = 0,1,2,..., \quad k = 0,1,2,...,r-1,$$

symmetrical with respect to the indices of their two multiindices $(i_1,...,i_r)$, $(j_1,...,j_k)$ and satisfying the conditions

$$C_{(r,k)} = C_{(k,r)}^{H_{r+k,k}}, \quad C_{(k,r)} = C_{(r,k)}^{B_{k+r,k}}.$$

D e f i n i t i o n. The mdm polynomial $P_r(x)$ (4) is called the basic orthogonal r-degree polynomial in $L_2(\rho,\Omega)$ if it is orthogonal to the homogeneous polynomials $1, x, x^2,..., x^{r-1}$, $x^k = {}^{0,0}(x^k)$:

$$\int_{\Omega} P_r(x)x^k\rho(x)dx \begin{cases} = 0, & k = 0,1,...,r-1, \\ \neq 0, & k = r. \end{cases} \tag{5}$$

D e f i n i t i o n. The sequences of the mdm polynomials $P_r(x)$ (4) and $Q_r(x)$ (2) are called the completely orthonormal in $L_2(\rho,\Omega)$ if the conditions (3), (5) are satisfied and the following condition

$$\int_{\Omega} Q_r(x)P_k(x)\rho(x)dx = \begin{cases} 0, & k = 0,1,...,r-1, \\ D_{(r,r)}, & k = r, \end{cases} \tag{6}$$

is satisfied too. Here $D_{(r,r)}$ is the $2r$-dimensional order n matrix with the following structure:

$$D_{(r,r)} = (d_{i_1,i_2,...,i_r,j_1,j_2,...,j_r}), \quad i_1,i_2,...,i_r,j_1,j_2,...,j_r = 1,2,...,n. \tag{7}$$

The elements of this matrix are defined by the expression

$$d_{i_1,i_2,...,i_r,j_1,j_2,...,j_r} = \begin{cases} r_1!r_2!...r_n!, & perm(i_1,i_2,...,i_r) = (j_1,j_2,...,j_r), \\ 0, & perm(i_1,i_2,...,i_r) \neq (j_1,j_2,...,j_r), \end{cases} \tag{8}$$

in which $perm(i_1,i_2,...,i_r)$ means any permute of the values of the indices $i_1,i_2,...,i_r$, $r_1 + r_2 + ... + r_n = r$, and r_k is the number of repetitions of the k-value, $k = 1,2,...,n$.

The matrix $D_{(r,r)}$ (7), (8) has such a useful property that for any q-dimensional matrix $C = (c_{i_1,i_2,...,i_{q-r},j_1,j_2,...,j_r})$ with $q \geq r$ symmetrical with respect to the indices $j_1,j_2,...,j_r$ the following equality is fulfilled [10]:

$${}^{0,r}(CD_{(r,r)}) = r!C.$$

Let us introduce the initial i-th order moments ν_{x^i} and the initial-central and central-initial $(i + j)$-th order moments $\nu_{x^i x_c^j}$, $\nu_{x_c^i x^j}$ of the weight function $\rho(x)$:

$$\nu_{x^0} = \int_{\Omega} \rho(x)dx,$$

$$\nu_{x^i x^j} = \nu_{x^{i+j}} = \int_\Omega x^{i+j} \rho(x) dx, \quad i + j = 1, 2, ..., \tag{9}$$

$$\nu_{x^i x_c^j} = \int_\Omega x^i \left(x^j - \nu_{x^j} \right) \rho(x) dx, \quad i + j = 1, 2, ..., \tag{10}$$

$$\nu_{x_c^i x^j} = \int_\Omega \left(x^i - \nu_{x^i} \right) x^j \rho(x) dx, \quad i + j = 1, 2, ... \tag{11}$$

The moments (10), (11) have the following properties:

$$\nu_{x^i x_c^j} = \nu_{x_c^i x^j} = \nu_{x_c^i x_c^j} = \nu_{x^{i+j}} - \nu_{x^i} \nu_{x^j}, \tag{12}$$

$$\nu_{x_c^i x_c^j} = (\nu_{x_c^j x_c^i})^{B_{iq+jq,iq}} = \nu_{x^{j+i}} - \nu_{x^j} \nu_{x^i},$$

where $B_{iq+jq,iq}$ is the transpose substitution of the type "forward" [10, A3, 16]. These properties are proved by calculation of the formulae (10), (11). The properties (12) allow us to use the following notations:

$$\mu_{x^i x^j} = \nu_{x^{i+j}} - \nu_{x^i} \nu_{x^j}, \quad \mu_{x^j x^i} = \nu_{x^{j+i}} - \nu_{x^j} \nu_{x^i},$$

$$\mu_{x^i x^j} = (\mu_{x^j x^i})^{B_{iq+jq,iq}}.$$

Let us introduce also the mutual moments

$$\nu_{y^i x^j} = \int_\Omega y^i(x) x^j \rho(x) dx \tag{13}$$

with the properties

$$\nu_{y^i x_c^j} = \nu_{y_c^i x^j} = \nu_{y_c^i x_c^j} = \int_\Omega y^i(x) \left(x^j - \nu_{x^j} \right) \rho(x) dx = \nu_{y^i x^j} - \nu_{y^i} \nu_{x^j} = \mu_{y^i x^j}.$$

The weight function $\rho(x)$ in the case $\nu_{x^0} = 1$ represents the probability density function of the some random vector ξ.

Theorem [10]. *If the sequences of the mdm polynomials $P_r(x)$ (4) and $Q_r(x)$ (2) are completely orthonormal in $L_2(\rho,\Omega)$, i .e. they satisfy by the conditions* (3), (5), (6)*, then the coefficients $C_{(r,k)}$ of the basic sequence $P_r(x)$ (4) are defined by the following mdm system of the linear algebraic equations*

$$\nu_{x^r x^p} + \sum_{k=0}^{r-1} {}^{0,k}(C_{(r,k)} \nu_{x^k x^p}) = 0, \quad r = 0, 1, ..., \quad p = 0, 1, ..., r-1, \tag{14}$$

and the coefficients $C^*_{(r,k)}$ of the conjugate sequence $Q_r(x)$ (2) are defined by the expression

$$C^*_{(r,k)} = r!^{\,0,r} \left({}^{0,r}B^{-1}_{(r,r)} C_{(r,k)} \right),$$

where ${}^{0,r}B^{-1}_{(r,r)}$ is the matrix (0,r)-inverse to the following matrix $B_{(r,r)}$:

$$B_{(r,r)} = \nu_{x^r x^r} + \sum_{k=0}^{r-1} {}^{0,k}(C_{(r,k)} \nu_{x^k x^r}) + \sum_{k=0}^{r-1} {}^{0,k}(\nu_{x^r x^k} C_{(k,r)}) + \sum_{k=0}^{r-1} \sum_{q=0}^{r-1} {}^{0,k}\left(C_{(r,k)} {}^{0,q}(\nu_{x^k x^q} C_{(q,r)}) \right). \tag{15}$$

The coefficients $C_{(k,r)}$ of the basic sequence $P_r(x)$ (4) are defined by the following mdm system of the linear algebraic equations

$$\nu_{x^p x^r} + \sum_{k=0}^{r-1} {}^{0,k}(\nu_{x^p x^k} C_{(k,r)}) = 0, \quad r = 0, 1, ..., \quad p = 0, 1, ..., r-1, \tag{16}$$

and the coefficients $C^*_{(k,r)}$ of the conjugate sequence $Q_r(x)$ (2) are defined by the expression

$$C^*_{(k,r)} = r!^{\,0,r}\left(C_{(k,r)}{}^{0,r}B^{-1}_{(r,r)}\right).$$

2. The Fourier series on the orthogonal polynomials. The Fourier series for the p-dimensional-matrix function $y(x)$ of the vector (one-dimensional-matrix) variable $x \in \Omega \subseteq R^n$ on the conjugate orthogonal polynomials $Q_r(x)$ (2) has the following form:

$$y(x) \sim \sum_{r=0}^{\infty} \frac{1}{r!}{}^{0,r}\left(B_r Q_r(x)\right), \tag{17}$$

where $B_r = (b_{j_1,j_2,\ldots,j_p,i_1,i_2,\ldots,i_r})$ are $(p + r)$-dimensional matrices of the n-order of the coefficients symmetrical when $r \geq 2$ with respect to the indices $i_1, i_2, \ldots, i_r$. They are defined by the expressions [10]

$$B_r = \int_\Omega {}^{0,0}\left(y(x)P_r(x)\right)\rho(x)dx. \tag{18}$$

Substitution of the polynomial $P_r(x)$ (4) into (18) gives the following expression for the coefficients B_r:

$$B_r = \int_\Omega {}^{0,0}\left(y(x)P_r(x)\right)\rho(x)dx = \int_\Omega {}^{0,0}\left(y(x)\left(x^r + \sum_{k=0}^{r-1}{}^{0,k}\left(x^k C_{(k,r)}\right)\right)\right)\rho(x)dx =$$

$$= E\left({}^{0,0}(yx^r) + \sum_{k=0}^{r-1}{}^{0,k}\left({}^{0,0}(yx^k)C_{(k,r)}\right)\right) = \nu_{yx^r} + \sum_{k=0}^{r-1}{}^{0,k}(\nu_{yx^k}C_{(k,r)}), \quad r = 0,1,2,\ldots, \tag{19}$$

where $E(\cdot)$ means the mathematical expectation.

The Fourier series on the basic orthogonal polynomials $P_r(x)$ (4) is obtained analogously:

$$y(x) \sim \sum_{r=0}^{\infty} \frac{1}{r!}{}^{0,r}\left(C_r P_r(x)\right), \tag{20}$$

where

$$C_r = \int_\Omega {}^{0,0}\left(y(x)Q_r(x)\right)\rho(x)dx.$$

Since

$$Q_r(x) = r!^{\,0,r}\left(P_r(x)^{0,r}B^{-1}_{(r,r)}\right),$$

then

$$C_r = \int_\Omega {}^{0,0}\left(y(x)Q_r(x)\right)\rho(x)dx = r!\int_\Omega {}^{0,0}\left(y(x)^{\,0,r}\left(P_r(x)^{0,r}B^{-1}_{(r,r)}\right)\right)\rho(x)dx =$$

$$= r!\int_\Omega {}^{0,r}\left({}^{0,0}\left(y(x)P_r(x)\right)^{0,r}B^{-1}_{(r,r)}\right)\rho(x)dx = r!^{\,0,r}\left(B_r{}^{0,r}B^{-1}_{(r,r)}\right). \tag{21}$$

The approximation of the scalar (zero-dimensional-matrix) function $y(\xi)$ of the random vector ξ with the probability density function $\rho(x)$ by the finite sum of the Fourier series

$$s_m(\xi) = \sum_{r=0}^{m} \frac{1}{r!}{}^{0,r}\left(B_r Q_r(\xi)\right) = \sum_{r=0}^{m} \frac{1}{r!}{}^{0,r}\left(C_r P_r(\xi)\right)$$

provides the minimum of the mean square error (m.s.e.) of the approximation

$$r_m^2 = E\left({}^{0,0}\left(y(\xi) - s_m(\xi)\right)^2\right) = \int_\Omega {}^{0,0}\left(y(x) - s_m(x)\right)^2 \rho(x)dx.$$

The minimal value $r^2_{m\min}$ of the m.s.e. is defined by the expression [10]

$$r^2_{m\min} = E\left(y^2(\xi)\right) - \sum_{r=0}^{m} \frac{1}{r!}{}^{0,r}(B_r C_r).$$

3. The polynomials orthogonal with the discrete weight function. The theory of the polynomials orthogonal with the continuous weight function $\rho(x)$ outlined above coincides with the theory of the polynomials orthogonal with the discrete weight function (p_k, x_k), when the l distinct points $x_1, x_2, \ldots, x_l$ are given in the region $\Omega \subseteq R^n$ with positive weights $p_1, p_2, \ldots, p_l$ and the measure μ of the region Ω is defined by the formula $\mu(\Omega) = \sum_{x_k \in \Omega} p_k$ [17]. They say in this case about the polynomials orthogonal on the system of the points. The moments (9) is defined in this case by the expression

$$\nu_{x^i x^j} = \nu_{x^{i+j}} = \int_{\Omega} x^{i+j} d\mu = \sum_{k=1}^{l} x_k^{i+j} p_k, \quad i+j = 1,2,\ldots,$$

and the mutual moments (13) is defined by the expression

$$\nu_{y^i x^j} = \int_{\Omega} y^i(x) x^j d\mu = \sum_{k=1}^{l} y_k^i x_k^j p_k, \quad i+j = 1,2,\ldots,$$

where $y_k = y(x_k)$, $k = 1,2,\ldots,l.$

We will call the discrete weight function with $\nu_{x^0} = 1$ as the discrete distribution of some random vector ξ. The important discrete distribution is so called empirical, or sample distribution, when x_i are the sample values of the random vector ξ and $p_i = 1/l$, where l is the length of the sample. If the empirical distribution is used then the approximation is called empirical.

4. The mdm approximation by the Fourier approximation. It is of interest to obtain the coefficients $c_{(m,k)}$ of the approximation of the function $y(x)$ by the mdm m degree polynomial

$$y(x) \sim \sum_{k=0}^{m} {}^{0,k}\left(c_{(p,k)} x^k\right) \tag{22}$$

in the case when the Fourier approximation (20) of this function of the same degree is obtained

$$y(x) \sim \sum_{k=0}^{m} \frac{1}{k!}{}^{0,k}\left(C_k P_k(x)\right). \tag{23}$$

The polynomial $P_k(x)$ of the fixed degree k provides in the expression (23) the following summand:

$$\frac{1}{k!}{}^{0,k}\left(C_k P_k(x)\right) = \frac{1}{k!}{}^{0,k}\left(C_k\left(\sum_{i=0}^{k} {}^{0,i}\left(C_{(k,i)} x^i\right)\right)\right) = \sum_{i=0}^{k} \frac{1}{k!}{}^{0,i}\left({}^{0,k}(C_k C_{(k,i)}) x^i\right). \tag{24}$$

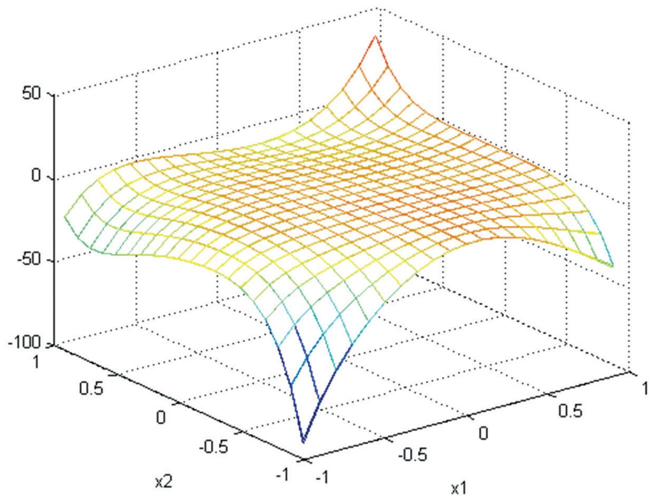
The variable x of the degree l, $l \le k$, appears in the expression (24) in the summand ${}^{0,l}\left({}^{0,k}(C_k C_{(k,l)}) x^l\right)/k!$. Summation of the coefficients at x^l by k from l to m gives the following formula for the desired coefficients:

$$c_{(p,l)} = \sum_{k=l}^{m} \frac{1}{k!}{}^{0,k}(C_k C_{(k,l)}), \quad l = 0,1,2,..,m. \tag{25}$$

If one takes in account that $C_{(i,i)} = E_s(0,i)$ is the symmetrical identity matrix which ensures the equality ${}^{0,i}(C_i C_{(i,i)}) = C_i$, then instead (25) we will have the expression

$$c_{(p,l)} = \frac{1}{l!}C_l + \sum_{k=l+1}^{m} \frac{1}{k!}{}^{0,k}(C_k C_{(k,l)}), \quad l = 0,1,2,..,m. \tag{26}$$

5. Computer simulation. The algorithm of the function approximation by the Fourier series was realized programmatically in the form of the standard Matlab function for general case of the theorem and was checked on many functions.



Real function and its two empirical approximations

We show the empirical approximation (according the p. 3) of the scalar function y of the two arguments x_1, x_2 as the polynomial (22) of the 7 degree (p = 0, m = 7). The scalar values of the coefficients $c_{(p,k)}$ of the polynomial are random integer from –5 to 5. The measurement errors are independent normal with zero mean and variation 0.2. The approximating polynomial has the degree 7 too.

We will call the approximation by the algorithm developed in this article as the Fourier mdm approximation (Fmdm-approximation) in opposite to the mdm approximation (mdm-approximation) of the work [18].

Figure shows three surfaces: real polynomial and its mdm-approximation and Fmdm-approximation. These surfaces are visually indistinguishable in Figure. The design of the experiment is random, the values of the variables x_1, x_2 are chosen from the uniform distribution U(–1,1). The number of runs is 255. Both of the methods have the high accuracy of the approximation. However, the program of the Fmdm-approximation turned out to be faster-acting compared to the program of the mdm-approximation. Other benefits are to be found out.

It should be noted that the classical approximation for the considered case is impossible because of its bulkiness and undeveloped.

The considered approximations have the undoubted advantages compared to the classical approach: algorithmical generality and extensive possibilities. However, they have the certain hardware limitations: out of memory and unacceptably long calculation time for the personal computer in the case of big data.

6. The orthogonal 0–2 degrees polynomials. In the work [10], the expressions of zero and first degree orthonormal polynomials and the particular cases of the second degree polynomials are obtained. These results are completed in this article by the general expressions of the second degree polynomials and Fourier series. The complete expressions are presented in the table for the case $\nu_{x^0} = 1$. The necessary proofs are given in the appendix.

Orthogonal polynomials and Fourier series up to second degree inclusive

Polynomials $P(x)$	Polynomials $Q(x)$
$P_0(x) = 1.$	$Q_0(x) = 1.$
$P_1(x) = C_{(1,0)} + x, \quad C_{(1,0)} = -\nu_x.$	$Q_1(x) = {}^{0,1}\left({}^{0,1}B_{(1,1)}^{-1} P_1(x)\right), \quad B_{(1,1)} = \mu_{xx} = \nu_{xx} - \nu_x \nu_x.$
$P_2(x) = C_{(2,0)} + {}^{0,1}(C_{(2,1)} x) + x^2,$ $C_{(2,1)} = -{}^{0,1}\left(\mu_{x^2 x}\, {}^{0,1}\mu_{xx}^{-1}\right),$ $C_{(2,0)} = -{}^{0,1}(C_{(2,1)} \nu_x) - \nu_{x^2},$ $\mu_{x^2 x} = \nu_{x^3} - \nu_{x^2} \nu_x.$	$Q_2(x) = 2!^{0.2}\left({}^{0,2}B_{(2,2)}^{-1} P_2(x)\right),$ $B_{(2,2)} = \mu_{x^2 x^2} - {}^{0,1}\left({}^{0,1}\left(\mu_{x^2 x}\, {}^{0,1}\mu_{xx}^{-1}\right)\mu_{xx^2}\right),$ $\mu_{x^2 x^2} = \nu_{x^4} - \nu_{x^2} \nu_{x^2},$ $\mu_{x^2 x} = \nu_{x^3} - \nu_{x^2} \nu_x.$

End of table

Fourier series on the polynomials $P(x)$	Fourier series on the polynomials $Q(x)$
$y(x) \sim {}^{0,0}\big(C_0 P_0(x)\big) + {}^{0,1}\big(C_1 P_1(x)\big) + \frac{1}{2}\,{}^{0,2}\big(C_2 P_2(x)\big),$ $C_0 = {}^{0,0}\left(B_0\,{}^{0,0}B^{-1}_{(0,0)}\right) = \nu_y,$ $C_1 = {}^{0,1}\left(B_1\,{}^{0,1}B^{-1}_{(1,1)}\right) = {}^{0,1}\left(\mu_{yx}\,{}^{0,1}\mu^{-1}_{xx}\right),$ $C_2 = 2\,{}^{0,2}\left(B_2\,{}^{0,2}B^{-1}_{(2,2)}\right).$	$y(x) \sim {}^{0,0}\big(B_0 Q_0(x)\big) + {}^{0,1}\big(B_1 Q_1(x)\big) + \frac{1}{2}\,{}^{0,2}\big(B_2 Q_2(x)\big),\ B_0 = \nu_y,$ $B_1 = \mu_{yx},$ $B_2 = \mu_{yx^2} - {}^{0,1}\left(\mu_{yx}\,{}^{0,1}\left({}^{0,1}\mu^{-1}_{xx}\,\mu_{x^2x}\right)\right).$

Conclusion. The known results of the Fourier series on the orthogonal polynomial are extended to the case of the mdm functions, what allows us to solve the new problems such as approximation of parametric curves and surfaces. The analytical expressions for the orthogonal polynomials and Fourier series of the second degree useful for the potential analytical studies are obtained. The theoretical results are realized as the single function of the programming language with many possibilities. Such a property we call the algorithmic generality. The efficiency of the program function is confirmed on the instance, performing of which is impossible by the classical approach.

Appendix. A1. Calculation of the polynomials of the small degrees. Let us obtain the orthonormal polynomials of the small degrees by solving the system of the equations (14).

We get the zero degree polynomials by definition:

$$P_0(x) = 1,$$

$$Q_0(x) = 1.$$

The one degree polynomials are obtained when $m = 1$ in the expressions (14). The system of the equations (14) consists of one equation:

$${}^{0,0}(C_{(1,0)}\nu_{x^0}) = -\nu_x.$$

If $\nu_{x^0} = 1,$ then

$$C_{(1,0)} = C_{(0,1)} = -\nu_x,$$

$$P_1(x) = x + {}^{0,0}\left(C_{(1,0)}x^0\right) = x - \nu_x.$$

The expression for the matrix $B_{(1,1)}$ from the expression (15) will look like this:

$$B_{(1,1)} = \nu_{xx} + {}^{0,0}(C_{(1,0)}\nu_x) + {}^{0,0}(\nu_x C_{(0,1)}) + {}^{0,0}\left(C_{(1,0)}(\nu_{x^0}C_{(0,1)})\right).$$

Taking into account the expressions for $C_{(0,1)}$ and $C_{(1,0)}$ when $\nu_{x^0} = 1$ we get

$$B_{(1,1)} = \nu_{xx} - \nu_x\nu_x = \mu_{xx}.$$

Then

$$Q_1(x) = {}^{0,1}\left({}^{0,1}B^{-1}_{(1,1)}P_1(x)\right) = {}^{0,1}\left({}^{0,1}\mu^{-1}_{xx}(x - \nu_x)\right).$$

The calculation of the first degree polynomials is completed.

The second degree polynomials are obtained when $m = 2$ in the expressions (14). We have now the following system, which consists of two equations (when $\nu_{x^0} = 1$):

$$C_{(2,0)} + {}^{0,1}(C_{(2,1)}\nu_x) = -\nu_{xx},$$

$${}^{0,0}(C_{(2,0)}\nu_x) + {}^{0,1}(C_{(2,1)}\nu_{xx}) = -\nu_{x^2x}.$$

We will solve this system by the Gauss elimination method. For this purpose, we subtract the first equation multiplied on the right by ν_x in the sense of the (0,0)-rolled product from the second equation. We will get the following system:

$$C_{(2,0)} + {}^{0,1}(C_{(2,1)}\nu_x) = -\nu_{xx},$$

$${}^{0,1}\left(C_{(2,1)}(\nu_{xx} - \nu_x\nu_x)\right) = -(\nu_{x^2x} - \nu_{xx}\nu_x),$$

or in other notation

$$C_{(2,0)} + {}^{0,1}(C_{(2,1)}\nu_x) = -\nu_{xx},$$

$${}^{0,1}(C_{(2,1)}\mu_{xx}) = -\mu_{x^2x}.$$

We get the expression for the coefficient $C_{(2,1)}$ from the second equation:

$$C_{(2,1)} = -\,{}^{0,1}\left(\mu_{x^2x}\,{}^{0,1}\mu_{xx}^{-1}\right).$$

Substituting this expression into first equation we get the expression for the coefficient $C_{(2,0)}$:

$$C_{(2,0)} = -\nu_{xx} - {}^{0,1}(C_{(2,1)}\nu_x) = -\nu_{xx} + {}^{0,1}\left({}^{0,1}(\mu_{x^2x}\,{}^{0,1}\mu_{xx}^{-1})\nu_x\right).$$

The coefficients $C_{(1,2)}$ and $C_{(0,2)}$ are obtained from the following system of the equations

$$C_{(0,2)} + {}^{0,1}(\nu_x C_{(1,2)}) = -\nu_{xx},$$

$${}^{0,0}(\nu_x C_{(0,2)}) + {}^{0,1}(\nu_{xx}C_{(1,2)}) = -\nu_{xx^2},$$

which follows from (16) when $m = 2$. Solving this system by Gauss elimination method we get

$$C_{(1,2)} = -\,{}^{0,1}\left({}^{0,1}\mu_{xx}^{-1}\mu_{xx^2}\right),$$

$$C_{(0,2)} = -\nu_{xx} - {}^{0,1}(\nu_x C_{(1,2)}) = -\nu_{xx} + {}^{0,1}\left(\nu_x\,{}^{0,1}({}^{0,1}\mu_{xx}^{-1}\mu_{xx^2})\right).$$

The second degree polynomial $P_2(x)$ of the basic sequence of the orthogonal polynomials has the form

$$P_2(x) = x^2 + {}^{0,1}(C_{(2,1)}x) + C_{(2,0)} = x^2 + {}^{0,1}(xC_{(1,2)}) + C_{(0,2)}.$$

The second degree polynomial $Q_2(x)$ of the conjugate sequence of the orthogonal polynomials is defined by the formula

$$Q_2(x) = 2!^{0,2}\left({}^{0,2}B_{(2,2)}^{-1}P_2(x)\right),$$

where, from (15),

$$B_{(2,2)} = \nu_{x^2x^2} + \sum_{k=0}^{1}{}^{0,k}(C_{(2,k)}\nu_{x^{k+2}}) + \sum_{k=0}^{1}{}^{0,k}(\nu_{x^{2+k}}C_{(k,2)}) + \sum_{k=0}^{1}\sum_{q=0}^{1}{}^{0,k}\left(C_{(2,k)}\,{}^{0,q}(\nu_{x^{k+q}}C_{(q,2)})\right).$$

Let us find the matrix $B_{(2,2)}$ for the case $\nu_{x^0} = 1$:

$$B_{(2,2)} = \nu_{x^2x^2} + \underline{\underline{{}^{0,0}(C_{(2,0)}\nu_{xx})}} + {}^{0,1}(C_{(2,1)}\nu_{xx^2}) + \underline{{}^{0,0}(\nu_{xx}C_{(0,2)})} + {}^{0,1}(\nu_{x^2x}C_{(1,2)}) +$$
$$+ {}^{0,0}(C_{(2,0)}C_{(0,2)}) + \underline{\underline{{}^{0,0}\left(C_{(2,0)}\,{}^{0,1}(\nu_x C_{(1,2)})\right)}} +$$
$$+ \underline{{}^{0,0}\left({}^{0,1}(C_{(2,1)}\nu_x)C_{(0,2)}\right)} + {}^{0,1}\left(C_{(2,1)}\,{}^{0,1}(\nu_{xx}C_{(1,2)})\right).$$

Combining the similar terms highlighted in the previous expression we get

$$B_{(2,2)} = \nu_{x^4} +{}^{0,0}\left(C_{(2,0)}\left(\nu_{xx} +{}^{0,1}(\nu_x C_{(1,2)})\right)\right) +{}^{0,0}\left(\left(\nu_{xx} +{}^{0,1}(C_{(2,1)}\nu_x)\right)C_{(0,2)}\right) +$$
$$+{}^{0,0}(C_{(2,0)}C_{(0,2)}) +{}^{0,1}(C_{(2,1)}\nu_{xx^2}) +{}^{0,1}(\nu_{x^2x}C_{(1,2)}) +{}^{0,1}\left(C_{(2,1)}{}^{0,1}(\nu_{xx}C_{(1,2)})\right) =$$
$$= e1 + (e2 + e3) + e4 + (e5 + e6) + e7 .$$

Taking into account the expressions $C_{(2,0)} = -{}^{0,1}(C_{(2,1)}\nu_x) - \nu_{xx}$, $C_{(0,2)} = -\nu_{xx} - {}^{0,1}(\nu_x C_{(1,2)})$ gives

$$e2 = -\underline{{}^{0,0}\left({}^{0,1}(C_{(2,1)}\nu_x)\nu_{xx}\right)} - \overline{{}^{0,0}\left({}^{0,1}(C_{(2,1)}\nu_x){}^{0,1}(\nu_x C_{(1,2)})\right)} - {}^{0,0}(\nu_{xx}\nu_{xx}) - \underline{\underline{{}^{0,0}\left(\nu_{xx}{}^{0,1}(\nu_x C_{(1,2)})\right)}},$$

$$e3 = -\underline{\underline{{}^{0,0}\left(\nu_{xx}{}^{0,1}(\nu_x C_{(1,2)})\right)}} - \overline{{}^{0,0}\left({}^{0,1}(C_{(2,1)}\nu_x){}^{0,1}(\nu_x C_{(1,2)})\right)} - {}^{0,0}(\nu_{xx}\nu_{xx}) - \underline{{}^{0,0}\left({}^{0,1}(C_{(2,1)}\nu_x)\nu_{xx}\right)},$$

$$e4 = \overline{{}^{0,1}\left(C_{(2,1)}{}^{0,1}\left((\nu_x\nu_x)C_{(1,2)}\right)\right)} + \underline{{}^{0,1}\left(C_{(2,1)}(\nu_x\nu_{xx})\right)} + \underline{\underline{{}^{0,1}\left((\nu_{xx}\nu_x)C_{(1,2)}\right)}} +{}^{0,0}(\nu_{xx}\nu_{xx}),$$

$$e5 = \underline{{}^{0,1}(C_{(2,1)}\nu_{xx^2})},$$

$$e6 = \underline{\underline{{}^{0,1}(\nu_{x^2x}C_{(1,2)})}},$$

$$e7 = \overline{{}^{0,1}\left(C_{(2,1)}{}^{0,1}(\nu_{xx}C_{(1,2)})\right)}.$$

Summation of the terms $e1$–$e7$ and combining the highlighted similar terms leads to the expression

$$B_{(2,2)} = \nu_{x^2x^2} + \underline{{}^{0,1}(C_{(2,1)}\nu_{xx^2})} - \underline{{}^{0,1}\left(C_{(2,1)}(\nu_x\nu_{xx})\right)} + \overline{{}^{0,1}\left(C_{(2,1)}{}^{0,1}(\nu_{xx}C_{(1,2)})\right)} -$$
$$-\overline{{}^{0,1}\left(C_{(2,1)}{}^{0,1}(\nu_x\nu_x C_{(1,2)})\right)} + \underline{\underline{{}^{0,1}(\nu_{x^2x}C_{(1,2)})}} - \underline{\underline{{}^{0,1}\left((\nu_{xx}\nu_x)C_{(1,2)}\right)}} - {}^{0,0}(\nu_{xx}\nu_{xx}),$$

or

$$B_{(2,2)} = \nu_{x^2x^2} - {}^{0,0}(\nu_{xx}\nu_{xx}) +{}^{0,1}\left(C_{(2,1)}(\nu_{xx^2} - \nu_x\nu_{xx})\right) +{}^{0,1}\left((\nu_{x^2x} - \nu_{xx}\nu_x)C_{(1,2)}\right) +$$
$$+{}^{0,1}\left(C_{(2,1)}{}^{0,1}\left((\nu_{xx} - \nu_x\nu_x)C_{(1,2)}\right)\right),$$

or

$$B_{(2,2)} = \mu_{x^2x^2} +{}^{0,1}(C_{(2,1)}\mu_{xx^2}) +{}^{0,1}(\mu_{x^2x}C_{(1,2)}) +{}^{0,1}\left(C_{(2,1)}{}^{0,1}(\mu_{xx}C_{(1,2)})\right).$$

Taking into account the expressions $C_{(2,1)} = -{}^{0,1}\left(\mu_{x^2x}{}^{0,1}\mu_{xx}^{-1}\right)$, $C_{(1,2)} = -{}^{0,1}\left({}^{0,1}\mu_{xx}^{-1}\,\mu_{xx^2}\right)$ we get

$$B_{(2,2)} = \mu_{x^2x^2} -{}^{0,1}\left({}^{0,1}(\mu_{x^2x}{}^{0,1}\mu_{xx}^{-1})\mu_{xx^2}\right) -{}^{0,1}\left(\mu_{x^2x}{}^{0,1}({}^{0,1}\mu_{xx}^{-1}\,\mu_{xx^2})\right) +$$
$$+{}^{0,1}\left(\mu_{x^2x}{}^{0,1}({}^{0,1}\mu_{xx}^{-1}\,\mu_{xx^2})\right),$$

or finally

$$B_{(2,2)} = \mu_{x^2x^2} -{}^{0,1}\left({}^{0,1}(\mu_{x^2x}{}^{0,1}\mu_{xx}^{-1})\mu_{xx^2}\right).$$

A2. Calculation of the Fourier series of the small degrees. The Fourier series (17) with three terms on the conjugate polynomials $Q_r(x)$ for the scalar function of the vector variable $y(x)$ has the form

$$y(x) \sim {}^{0,0}\left(B_0 Q_0(x)\right) + {}^{0,1}\left(B_1 Q_1(x)\right) + \frac{1}{2}\,{}^{0,2}\left(B_2 Q_2(x)\right).$$

Let us find the coefficients B_i of this series by the formula (19).

$$B_0 = \int_\Omega {}^{0,0}\left(y(x)P_0(x)\right)\rho(x)dx = \int_\Omega y(x)\rho(x)dx = \nu_y,$$

$$B_1 = \int_\Omega {}^{0,0}\left(y(x)P_1(x)\right)\rho(x)dx = \int_\Omega {}^{0,0}\left(y(x)(x-\nu_x)\right)\rho(x)dx = \mu_{yx},$$

$$B_2 = \int_\Omega {}^{0,0}\left(y(x)P_2(x)\right)\rho(x)dx = \int_\Omega {}^{0,0}\left(y(x)\left(x^2 + {}^{0,1}(xC_{(1,2)}) + C_{(0,2)}\right)\right)\rho(x)dx =$$

$$= \nu_{yx^2} + {}^{0,1}(\nu_{yx}C_{(1,2)}) + \nu_y C_{(0,2)} =$$

$$= \nu_{yx^2} - {}^{0,1}\left(\nu_{yx}\,{}^{0,1}\left({}^{0,1}\mu_{xx}^{-1}\mu_{xx^2}\right)\right) - {}^{0,1}\left((\nu_y\nu_x)C_{(1,2)}\right) - \nu_y\nu_{xx} =$$

$$= \nu_{yx^2} - {}^{0,1}\left(\nu_{yx}\,{}^{0,1}\left({}^{0,1}\mu_{xx}^{-1}\mu_{xx^2}\right)\right) + {}^{0,1}\left((\nu_y\nu_x)\,{}^{0,1}\left({}^{0,1}\mu_{xx}^{-1}\mu_{xx^2}\right)\right) - \nu_y\nu_{xx} =$$

$$= \mu_{yx^2} - {}^{0,1}\left(\mu_{yx}\,{}^{0,1}\left({}^{0,1}\mu_{xx}^{-1}\mu_{xx^2}\right)\right).$$

The Fourier series (17) with three terms on the basic polynomials $P_r(x)$ for the scalar function of the vector variable $y(x)$ has the form

$$y(x) \sim {}^{0,0}\left(C_0P_0(x)\right) + {}^{0,1}\left(C_1P_1(x)\right) + \frac{1}{2}\,{}^{0,2}\left(C_2P_2(x)\right).$$

We get in accordance with the formula $C_r = r!\,{}^{0,r}\left(B_r\,{}^{0,r}B_{(r,r)}^{-1}\right)$ (21):

$$C_0 = {}^{0,0}\left(B_0\,{}^{0,0}B_{(0,0)}^{-1}\right) = \nu_y,$$

$$C_1 = {}^{0,1}\left(B_1\,{}^{0,1}B_{(1,1)}^{-1}\right) = {}^{0,1}\left(\mu_{yx}\,{}^{0,1}\mu_{xx}^{-1}\right),$$

$$C_2 = 2\,{}^{0,2}\left(B_2\,{}^{0,2}B_{(2,2)}^{-1}\right).$$

The coefficients of the approximation of the function $y(x)$ by the series (22) on the degrees of the variable x up to second degree inclusive ($m = 2$) are defined by the following expressions defined by the formula (26):

$$c_{(p,0)} = \frac{1}{0!}C_0 + \sum_{k=1}^{2}\frac{1}{k!}\,{}^{0,k}(C_kC_{(k,0)}) = C_0 + {}^{0,1}(C_1C_{(1,0)}) + \frac{1}{2!}\,{}^{0,2}(C_2C_{(2,0)}),$$

$$c_{(p,1)} = \frac{1}{1!}C_1 + \sum_{k=2}^{2}\frac{1}{k!}\,{}^{0,k}(C_kC_{(k,1)}) = C_1 + \frac{1}{2!}\,{}^{0,2}(C_2C_{(2,1)}),$$

$$c_{(p,2)} = \frac{1}{2!}C_2.$$

A3. The main definitions of the multidimensional matrices [8, 10].

The definition of a multidimensional matrix. A multidimensional (p-dimensional) matrix is a system of numbers or variables $a_{i_1,i_2,\dots,i_p}$, $i_\alpha = 1,2,\dots,n_\alpha$, $\alpha = 1,2,\dots,p$, located at the points of the p-dimensional space defined by the coordinates $i_1, i_2, \dots, i_p$.

The p-dimensional matrix is denoted as

$$A = (a_{i_1,i_2,\dots,i_p}), \quad i_\alpha = 1,2,\dots,n_\alpha, \quad \alpha = 1,2,\dots,p,$$

or $A = (a_i)$, where $i = (i_1, i_2, \dots, i_p)$ is a multi-index, $i_\alpha = 1,2,\dots,n_\alpha$, $\alpha = 1,2,\dots,p$.

Thus, a zero-dimensional matrix is a scalar, a one-dimensional matrix is a vector and a two-dimensional matrix is an ordinary matrix in traditional notation.

Addition of multidimensional matrices. If $A=(a_{i_1,i_2,\dots,i_p})$, $B=(b_{i_1,i_2,\dots,i_p})$, then $C=A+B=(c_{i_1,i_2,\dots,i_p})$, where $c_{i_1,i_2,\dots,i_p}=a_{i_1,i_2,\dots,i_p}+b_{i_1,i_2,\dots,i_p}$, $i_\alpha=1,2,\dots,n_\alpha$, $\alpha=1,2,\dots,p.$

Multiplication of multidimensional matrix by a scalar. If t is some scalar number or a variable and A is a p-dimensional matrix, then $C=tA=(c_{i_1,i_2,\dots,i_p})$, where $c_{i_1,i_2,\dots,i_p}=ta_{i_1,i_2,\dots,i_p}$, $i_\alpha=1,2,\dots,n_\alpha$, $\alpha=1,2,\dots,p.$

Multiplication of two multidimensional matrices. If a p-dimensional matrix A is represented in the form of $A=(a_{i_1,i_2,\dots,i_p})=(a_{l,s,c})$, where $l=(l_1,l_2,\dots,l_\kappa)$, $s=(s_1,s_2,\dots,s_\lambda)$, $c=(c_1,\dots,c_\mu)$ are multi-indices, $\kappa+\lambda+\mu=p$, and a q-dimensional matrix B is represented in the form of $B=(b_{j_1,j_2,\dots,j_q})=(b_{c,s,m})$, where $m=(m_1,\dots,m_\nu)$ is a multi-index, $\lambda+\mu+\nu=q$, then the matrix $D=(d_{l,s,m})$ is called a (λ,μ)-folded product of the matrices A and B, if its elements are defined by the expression

$$d_{l,s,m}=\sum_c a_{l,s,c}b_{c,s,m}=\sum_{c_1}\sum_{c_2}\cdots\sum_{c_\mu}a_{l,s,c}b_{c,s,m}.$$

The (λ,μ)-folded product of the matrices A and B is denoted ${}^{\lambda,\mu}(AB)$. Thus

$$D={}^{\lambda,\mu}(AB)=\left(\sum_c a_{l,s,c}b_{c,s,m}\right)=(d_{l,s,m}).$$

In the case of the (0,0)-folded product we often omit the left upper indices and write AB instead of ${}^{0,0}(AB)$.

The degree of multidimensional matrix. The matrix $D={}^{\lambda,\mu}(AA)={}^{\lambda,\mu}A^2$ is called a (λ,μ)-folded square of the matrix A, and the matrix $D={}^{\lambda,\mu}\left(A\,{}^{\lambda,\mu}\left(A\cdots{}^{\lambda,\mu}(AA)\right)\right)={}^{\lambda,\mu}A^k$ is called a (λ,μ)-folded k-th degree of the matrix A. If it is (0,0)-folded k-th degree of the matrix A, then we omit the left upper indices and write A^k instead of ${}^{0,0}A^k$.

A multidimensional identity matrix. The matrix $E(\lambda,\mu)$ is called a (λ,μ)-identity matrix if for any multidimensional matrix A the equality

$${}^{\lambda,\mu}\left(AE(\lambda,\mu)\right)={}^{\lambda,\mu}\left(E(\lambda,\mu)A\right)=A$$

is fulfilled. $E(\lambda,\mu)$ is $(\lambda+2\mu)$-dimensional matrix whose elements are defined by the formula

$$E(\lambda,\mu)=(e_{c,s,m})=\left(\begin{cases}1, & if\ \ c=m,\\ 0, & if\ \ c\neq m\end{cases}\right),$$

$$c=(c_1,\dots,c_\mu),\quad s=(s_1,s_2,\dots,s_\lambda),\quad m=(m_1,\dots,m_\mu).$$

A multidimensional inverse matrix. The matrix $A^{-1}(\lambda,\mu)$ (or ${}^{\lambda,\mu}A^{-1}$) is called a (λ,μ)-inverse to the matrix A, if the equalities

$${}^{\lambda,\mu}\left(AA^{-1}(\lambda,\mu)\right)={}^{\lambda,\mu}\left(A^{-1}(\lambda,\mu)A\right)=E(\lambda,\mu)$$

are satisfied.

The transpose of a multidimensional matrix. The matrix $A^T=\left(a^T_{i_1,i_2,\dots,i_p}\right)$ the elements $a^T_{i_1,i_2,\dots,i_p}$ of which are connected with the elements $a_{i_1,i_2,\dots,i_p}$ of the matrix $A=(a_{i_1,i_2,\dots,i_p})$ by the equalities

$$a^T_{i_1,i_2,\dots,i_p}=a_{i_{\alpha_1},i_{\alpha_2},\dots,i_{\alpha_p}}, \tag{A.3.1}$$

where $i_{\alpha_1},i_{\alpha_2},\dots,i_{\alpha_p}$ is some permutation of the indices $i_1,i_2,\dots,i_p$ is called the transposed according to the substitution $T=\begin{pmatrix}i_1,\dots,\ i_p\\ i_{\alpha_1},\dots,i_{\alpha_p}\end{pmatrix}$ matrix A.

In the work [10] some standard substitutions are introduced that allow us to form various substitutions, two of which of the types ‘forward’, ‘back’.

The substitution on the p indices the lower string of which is formed from the upper string by the transfer of the r left indices to the right (forward) is called substitution of the type ‘forward’:

$$B_{p,r} = \begin{pmatrix} i_1, & i_2, & \dots, & i_{p-r}, & i_{p-r+1}, & \dots, & i_p \\ i_{r+1}, & i_{r+2}, & \dots, & i_p, & i_1, & \dots, & i_r \end{pmatrix}, \quad p \geq r.$$

The substitution on the p indices the lower string of which is formed from the upper string by the transfer of the r right indices to the left (back) is called substitution of the type ‘back’:

$$H_{p,r} = \begin{pmatrix} i_1, & i_2, & \dots, & i_r, & i_{r+1}, & \dots, & i_p \\ i_{p-r+1}, & i_{p-r+2}, & \dots, & i_p, & i_1, & \dots, & i_{p-r} \end{pmatrix}, \quad p \geq r.$$

The Matlab’s function ipermute.m performs a transpose of a multidimensional array in accordance with the definition (A.3.1).

Matrices associated with multidimensional matrices. Let the p-dimensional matrix $A = (a_{i_1, i_2, \dots, i_p})$ of the order n be represented in the form of $A = (a_{l,s,c})$, where $l = (l_1, l_2, \dots, l_\kappa)$, $s = (s_1, s_2, \dots, s_\lambda)$, $c = (c_1, \dots, c_\mu)$ are multi-indices, $\kappa + \lambda + \mu = p$. In this case we say that the matrix A have the (κ,λ,μ)-structure and is denoted $A_{(\kappa,\lambda,\mu)}$. The multi-indices l, s, c of this matrix have n^κ, n^λ and n^μ values respectively. Let us arrange the values of l, s, c in some way: $\tilde{l} = l^{(1)}, l^{(2)}, \dots, l^{(n^\kappa)}$, $\tilde{s} = s^{(1)}, s^{(2)}, \dots, s^{(n^\lambda)}$, $\tilde{c} = c^{(1)}, c^{(2)}, \dots, c^{(n^\mu)}$. The block-diagonal matrix

$$\tilde{A}_{(\kappa,\lambda,\mu)} = \operatorname{diag}\left\{ A^{(1)}_{(\kappa,0,\mu)}, A^{(2)}_{(\kappa,0,\mu)}, \dots, A^{(n^\lambda)}_{(\kappa,0,\mu)} \right\},$$

consisting of elements of the matrix A, where the diagonal blocks $A^{(h)}_{(\kappa,0,\mu)}$, $h = 1, 2, \dots, n^\lambda$, are two-dimensional $(n^\kappa \times n^\mu)$-matrices

$$A^{(h)}_{(\kappa,0,\mu)} = (a_{\tilde{l}, s^{(h)}, \tilde{c}}), \quad \tilde{l} = l^{(1)}, l^{(2)}, \dots l^{(n^\kappa)}, \quad \tilde{c} = c^{(1)}, c^{(2)}, \dots, c^{(n^\mu)},$$

is called (κ,λ,μ)-associated with the matrix $A_{(\kappa,\lambda,\mu)}$.

The associated matrix $\tilde{A}_{(\kappa,\lambda,\mu)}$ represented completely the initial multidimensional matrix $A_{(\kappa,\lambda,\mu)}$, because it contains all its elements.

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