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CLASSICAL SOLUTION TO MIXED PROBLEMS FROM THE THEORY OF LONGITUDINAL IMPACT ON AN ELASTIC SEMI-INFINITE ROD IN THE CASE OF SEPARATION OF THE IMPACTING BODY AFTER THE COLLISION

Abstract. In this work, we consider two coupled initial-boundary value problems, which, based on the Saint-Venant theory, model the longitudinal impact phenomena in a semi-infinite rod. The mathematical formulation of the problem is two mixed problems for the one-dimensional wave equation with conjugation conditions. The Cauchy conditions are specified on the spatial half-line. The initial condition for the partial derivative with respect to the time variable has a discontinuity of the first kind at one point. The boundary condition, which includes the unknown function and its first- and second-order partial derivatives, is specified on the time half-line. The solution is constructed by the method of characteristics in an explicit analytical form. The uniqueness of the solution is proved, and the conditions under which a piecewise-smooth solution exists are established. The classical solution to a mixed problem with matching conditions is considered.

Keywords: longitudinal impact, wave equation, mixed problem, classical solution, method of characteristics, discontinuous initial conditions, discontinuous boundary conditions, matching conditions, conjugation conditions

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КЛАССИЧЕСКОЕ РЕШЕНИЕ СМЕШАННЫХ ЗАДАЧ ИЗ ТЕОРИИ ПРОДОЛЬНОГО УДАРА ПО УПРУГОМУ ПОЛУБЕСКОНЕЧНОМУ СТЕРЖНЮ В СЛУЧАЕ ОТДЕЛЕНИЯ УДАРИВШЕГО ТЕЛА ПОСЛЕ УДАРА

Аннотация. Рассматриваются две связанные начально-краевые задачи, которые моделируют процесс продольного удара в полубесконечном стержне на основе теории Сен-Венана. Математическая постановка задачи представляет собой две смешанные задачи для одномерного волнового уравнения с условиями сопряжения. Условия Коши задаются на пространственной полупрямой. Начальное условие для частной производной по временной переменной имеет разрыв первого рода в одной точке. На временной полупрямой задается граничное условие, содержащее неизвестную функцию и ее частные производные первого и второго порядка. Решение строится методом характеристик в явном аналитическом виде. Доказана единственность и установлены условия существования кусочно-гладкого решения. Рассмотрено классическое решение смешанной задачи с условиями сопряжения.

Ключевые слова: продольный удар, волновое уравнение, смешанная задача, классическое решение, метод характеристик, разрывные начальные условия, разрывные граничные условия, условия согласования, условия сопряжения

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Introduction. Impact mechanics is a subject that deals with the reaction forces that develop during a collision and the dynamic response of structures to these reaction forces. It has a wide range of engineering applications, from designing sports equipment to improving the crashworthiness of automobiles [1]. The longitudinal impact on elastic rods is one of the classical problems of this theory. This problem has been considered by both old researchers [2–4] and newer ones [5–15].

This paper is devoted to the mathematical study of two coupled mixed problems from the theory of longitudinal impact using the classical method of characteristics. The method of characteristics, despite its age, is one of the most important in solving problems from the theory of longitudinal impact. Unlike the method of separation of variables or the contour integral method, it allows the construction of solutions in an explicit analytical form, good for programming, see, for example, the paper [16] and the dissertation [17]. The main difficulty of studying mixed problems from the theory of longitudinal impact is the fact that these mixed problems contain conditions with discontinuous functions. This work is a continuation of the recent paper [18]. The papers [19–21] are close to the present work.

1. Statement of the problem. Suppose that an elastic semi-infinite homogeneous rod of constant cross-section, whose end $x = 0$ is elastically fixed, is subjected at the initial moment $t = 0$ to an impact on the end $x = 0$ by a load that sticks to the rod and separates from the rod at the moment $t = T$. We also assume that an external volumetric force acts on the rod and that the displacements of the rod and the rate of their change at the initial moment $t = 0$ are not equal to zero. Then, neglecting both the weight of the rod as a force and its possible vertical displacements, to study the vibrations of the rod, we have to find solutions to two coupled mixed problems:

1) in the domain $Q_1 = (0,\infty)\times(0,T),$ we have to find a solution to the equation

$$\left(\partial_t^2 - a_1^2\partial_x^2\right)u_1(t,x) = f_1(t,x), \tag{1}$$

with the initial conditions

$$u_1(0,x) = \varphi(x), \quad x \ge 0, \quad \partial_t u_1(0,x) = \psi(x) = \begin{cases} \psi_1, & x = 0, \\ \psi_2(x), & x < 0, \end{cases} \tag{2}$$

and the boundary condition

$$\left(\partial_t^2 - b^2\partial_x + c^2\right)u_1(t,0) = \begin{cases} \mu_0, & t = 0, \\ \mu_1(t), & 0 < t \le T; \end{cases} \tag{3}$$

2) in the domain $Q_2 = (0,\infty)\times(T,\infty),$ we have to find a solution to the equation

$$\left(\partial_t^2 - a_2^2\partial_x^2\right)u_2(t,x) = f_2(t,x), \tag{4}$$

with the initial conditions

$$u_2(t = T,x) = u_1(T,x), \quad \partial_t u_2(t = T,x) = \partial_t u_1(T,x), \quad x \ge 0, \tag{5}$$

and the boundary condition

$$(h - \partial_x)u_2(t,0) = \mu_2(t), \quad t > T. \tag{6}$$

The relations (1)–(6) use the following notation: $a_1^2 = a_2^2 = E/\rho,$ $b^2 = SE/M,$ $c^2 = k/M,$ $h = k/ES$, where $E > 0$ is Young's modulus of the rod material, $\rho = 0$ is the density of the rod material, $S = 0$ is the cross-sectional area of the rod, $M > 0$ is the mass of the impacted load, $k > 0$ is the stiffness coefficient of the linear elastic element to which the end $x = 0$ of the rod is attached. The quantity $\psi_1 - \psi_2(0+)$ has a physical meaning of the velocity of the impacting load, $\mu_1(t)$ has a physical meaning of the external force acting on the end of the rod, divided by the mass of the impacting load, and $\mu_2(t)$ has a physical meaning of the external force acting on the end of the rod, divided by ES. The value μ_0 is not an arbitrary preset constant [6], but it depends on the functions f, φ, ψ and will be determined later.

We will assume that $a_1 > 0$ and $a_2 > 0$ for definiteness. We also note that mathematically, the signs of h, b^2, and c^2 do not affect the correctness of the problem. And despite that $h > 0$, $b^2 > 0$, and c^2 from physical assumptions, we will consider the problems in a general form, regardless of the signs of h, b^2, and c^2. Also, without loss of generality, we will assume that the numbers a_1 and a_2 are not necessarily equal.

2. Auxiliary problem. To construct a solution to problem (4)–(6), we consider an auxiliary mixed problem for the function v.

Statement of the problem. In the domain $Q = (0,\infty)\times(0,\infty)$ of two independent variables $(t,x)\in Q\subset\mathbb{R}^2$, we consider the wave equation

$$\left(\partial_t^2 - a^2\partial_x^2\right)v(t,x) = f(t,x), \tag{7}$$

with the initial conditions

$$v(0,x)=\tilde{\phi}(x)=\begin{cases}\tilde{\phi}_1(x), & x\in[0,x^*),\\ \tilde{\phi}(x^*), & x=x^*,\\ \tilde{\phi}_2(x), & x\in(x^*,\infty),\end{cases}\qquad \partial_t v(0,x)=\tilde{\psi}(x)=\begin{cases}\tilde{\psi}_1(x), & x\in[0,x^*),\\ \tilde{\psi}_2(x), & x\in(x^*,\infty),\end{cases} \tag{8}$$

and the boundary condition

$$(h-\partial_x)v(t,0)=\tilde{\mu}(t)=\begin{cases}\tilde{\mu}_1(t), & t<x^*/a,\\ \tilde{\mu}_2(t), & t>x^*/a,\end{cases} \tag{9}$$

where $\tilde{\phi}(x^*)=\tilde{\phi}_1(x^*-0)=\tilde{\phi}_2(x^*+0)$. Also, for definiteness, we assume $a > 0$.

Definition of the solution. Due to the discontinuous initial condition for the time derivative, the problem (7)–(9) has no classical solution defined on the set $\overline{Q}$. However, we can define a classical solution to the problem (7)–(9) on a smaller set $\overline{Q}/\Gamma$ such that it belongs to the class $C^2(\overline{Q}/\Gamma)$ and satisfies the equation (7), the initial conditions (8), the boundary condition (9), and additional matching conditions given on the set Γ.

Definition 1. *A continuous function v is called a classical solution to the problem* (7)–(9)*, if the following conditions are fulfilled: 1) the function v is twice continuously differentiable and satisfies Eq.* (7) *everywhere, except the characteristics* $x - at = 0$, $x - at = \pm x^*$ *and* $x + at = x^*$*; 2) the first initial condition* $v(0,x)=\tilde{\phi}(x)$ *is satisfied on the entire half-line* $x \geq 0$*; 3) the second initial condition* $\partial_t v(0,x)=\tilde{\psi}(x)$ *is satisfied on the set* $[0,x^*)\cup(x^*,\infty)$*; 4) the boundary condition* (9) *holds on the set* $[0,a^{-1}x^*)\cup(a^{-1}x^*,\infty)$*; 5) the function v satisfies the following conjugation conditions:*

$$[(v)^+ - (v)^-](t,at+x^*) = [(v)^+ - (v)^-](t,x^*-at) = 0, \tag{10}$$

$$[(v)^+ - (v)^-](t,at) = 0, \tag{11}$$

$$[(v)^+ - (v)^-](t,at-x^*) = 0 \tag{12}$$

on the characteristics $x-at=0$, $x-at=\pm x^*$ *and* $x+at=x^*$.

Remark 1. The system of conditions imposed on the function v in Definition 1 is overdetermined. For example, we can delete the word "*continuous*" at the beginning of Definition 1 because the continuity of the function v in this case follows from the facts that the function v is twice continuously differentiable everywhere, except the characteristics $x - at = 0$, $x - at = \pm x^*$ and $x + at = x^*$, and satisfies the conjugation conditions (10)–(12). Or vice versa, we can remove the requirement "*the function v satisfies the conjugation conditions* (10)–(12) *on the characteristics* $x - at = 0$, $x - at = \pm x^*$ *and* $x + at = x^*$", since the conjugation conditions follow from the continuity of the function v. The disadvantage of the second approach is that the main point in definitions solutions to problems with discontinuous data is the specification of the correct matching conditions, and, therefore, we should write out them explicitly.

Construction of the solution. Here, in contrast to [21], we will formally find a solution using the method of characteristics. It is known that the general solution of an inhomogeneous linear equation is the sum of the general solution of a homogeneous linear equation and a particular solution of the inhomogeneous one [22]. Let $w:\overline{Q}\to\mathbb{R}$ be a particular solution of the inhomogeneous wave equation (7) that satisfies the homogeneous Cauchy conditions $w(0,x)=\partial_t w(0,x)=0$ and $\partial_t^2 w(0,x)=f(0,x)$. Such a solution w exists, and it has the form [23]

$$w(t,x)=\frac{1}{2a}\int_0^t d\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)} f\left(\tau,|\xi|\right)d\xi. \tag{13}$$

If $f\in C^1(\overline{Q})$, then $w\in C^2(\overline{Q})$.

Thus, we can write the general solution of Eq. (7) in the form

$$v(t,x)=w(t,x)+G^{(1)}(x-at)+G^{(2)}(x+at). \tag{14}$$

Satisfying the Cauchy conditions in the subdomains $Q^{(1)}$ and $Q^{(2)}$, we obtain the formulas

$$\begin{aligned} G^{(1)}(x)&=\frac{\tilde{\phi}(x)}{2}-\frac{1}{2a}\int_0^x\tilde{\psi}(\xi)d\xi+C_1, \quad x\in(0,\infty),\\ G^{(2)}(x)&=\frac{\tilde{\phi}(x)}{2}+\frac{1}{2a}\int_0^x\tilde{\psi}(\xi)d\xi-C_1, \quad x\in(0,\infty), \end{aligned} \tag{15}$$

where C_1 is an arbitrary real constant.

Due to the representation (14), it is necessary to determine the function $G^{(1)}$ for all real numbers to solve the auxiliary problem (7)–(9). We have already done this for non-negative numbers according to the formula (15). To find the value of the function $G^{(1)}$ for negative values of the argument, we substitute the relation (14) into the boundary condition (9) and get

$$h\left(g^{(1,4)}(-at)+g^{(2,4)}(at)+w^*(0,t)\right)-Dg^{(1,4)}(-at)-Dg^{(2,4)}(at)-\partial_x w(0,t)=\mu_1(t),\ \ t\in(0,x^*/a).$$

From the previous equality we have an ordinary differential equation for the function $G^{(1)}$

$$\begin{aligned} &h\left(G^{(1)}(z)+G^{(2)}(-z)+w\left(0,-z/a\right)\right)-D\left[G^{(1)}\right](z)-\\ &-D\left[G^{(2)}\right](-z)-\partial_x w(0,-z/a)=\mu_1(t), \quad z\in(-x^*,0). \end{aligned} \tag{16}$$

We consider Eq. (16) as a differential equation with respect to the function $G^{(1)}$ on the half-line $z\in(-\infty,0]$. Let the condition

$$G^{(1)}(0_-)=G^{(1)}(0_+)=C_1+\frac{\tilde{\phi}_1(0)}{2} \tag{17}$$

also be satisfied. We consider Eq. (4.17) with respect to $G^{(1)}$ with the condition (17) as the Cauchy problem for the first-order differential equation. Solving this problem, we get

$$G^{(1)}(z)=\exp(hz)\left(C_1+\frac{\tilde{\phi}_1(0)}{2}+\int_0^z\exp(-h\eta)\mathcal{M}(\eta)d\eta\right), \quad z\le 0,$$

where

$$\mathcal{M}(z)=\tilde{\mu}\left(-\frac{z}{a}\right)-h\left(G^{(2)}(-z)+w\left(-\frac{z}{a},0\right)\right)+D\left[G^{(2)}\right](-z)+\partial_x w\left(-\frac{z}{a},0\right).$$

After simple transformations, taking into account the obvious equality

$$\int_0^z -\frac{1}{2}\exp(-h\eta)\phi'(-\eta)d\eta = \int_0^z \frac{h}{2}\exp(-h\eta)\phi(-\eta)\mathrm{d}\eta - \frac{\phi(0)}{2} + \frac{e^{-hz}}{2}\phi(-z),$$

we can write the expression $G^{(1)}(z)$, where $z \le 0$, in the following form

$$G^{(1)}(z) = \tilde{N}_1 + \frac{\tilde{\phi}(-z)}{2} + \frac{h\exp(hz)}{2}\int_0^z \exp(-h\eta)\tilde{\phi}(-\eta)d\eta + \frac{\exp(hz)}{2a}\times$$
$$\times\int_0^z \exp(-h\eta)\left[h\int_0^{-\eta}\tilde{\psi}(\xi)\mathrm{d}\xi + ah\left(2w\left(-\frac{\eta}{a},0\right) + \tilde{\phi}(-\eta)\right) - \right.$$
$$\left. -\tilde{\psi}(-\eta) - 2a\left(\tilde{\mu}\left(-\frac{\eta}{a}\right) + \partial_x w\left(-\frac{\eta}{a},0\right)\right)\right]d\eta, \quad z \le 0. \tag{18}$$

Substituting (15) and (18) into (14), we find out

$$v(t,x) = w(t,x) + \frac{\tilde{\phi}(x+at)+\tilde{\phi}(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\tilde{\psi}(\xi)d\xi, \quad t \ge 0, \quad x \ge 0, \quad x - at \ge 0,$$

$$v(t,x) = \frac{\tilde{\phi}(x+at)+\tilde{\phi}(x-at)}{2} + \frac{1}{2a}\int_0^{x+at}\tilde{\psi}(\xi)d\xi +$$
$$+\frac{h\exp\big(h(x-at)\big)}{2}\int_0^{x-at}\exp(-h\eta)\tilde{\phi}(-\eta)d\eta + \frac{\exp\big(h(x-at)\big)}{2a}\times$$
$$\times\int_0^{x-at}\exp(-h\eta)\left[h\int_0^{-\eta}\tilde{\psi}(\xi)\mathrm{d}\xi + ah\left(2w\left(-\frac{\eta}{a},0\right) + \tilde{\phi}(-\eta)\right) - \right.$$
$$\left. -\tilde{\psi}(-\eta) - 2a\left(\tilde{\mu}\left(-\frac{\eta}{a}\right) + \partial_x w\left(-\frac{\eta}{a},0\right)\right)\right]d\eta, \quad t \ge 0, \quad x \ge 0, \quad x - at \le 0. \tag{19}$$

Thus, we have constructed a formal piecewise smooth solution to the problem (7)–(9) determined by the formulas (13) and (19). By direct verification, we establish that if the following smoothness conditions are satisfied

$$f \in C^1(\bar{Q}), \quad \tilde{\phi}_1 \in C^2\left([0,x^*]\right), \quad \tilde{\phi}_2 \in C^2\left([x^*,\infty)\right), \quad \tilde{\psi}_1 \in C^1\left([0,x^*]\right),$$
$$\tilde{\psi}_2 \in C^1\left([x^*,\infty)\right), \quad \tilde{\mu}_1 \in C^1\left([0,a^{-1}x^*]\right), \quad \tilde{\mu}_2 \in C^1\left([a^{-1}x^*,\infty)\right),$$

then the function v determined by the expressions (13) and (19) is continuous, satisfies the wave equation (7) everywhere, except the characteristics $x - at = 0$, $x - at = \pm x^*$ and $x + at = x^*$, the first initial condition $v(0,x) = \tilde{\phi}(x)$ on the half-line $x \ge 0$, the second initial condition $\partial_t v(0,x) = \tilde{\psi}(x)$ on the set $[0,x^*)\cup(x^*,\infty)$ and the boundary condition (9) on the set $[0,a^{-1}x^*)\cup(a^{-1}x^*,\infty)$. It means that we have constructed the solution to the mixed problem (7)–(9) in the sense of Definition 1.

Uniqueness of the solution. The following theorem holds.

T h e o r e m 1. *Let the smoothness conditions*

$$f \in C^1(\bar{Q}), \quad \tilde{\phi}_1 \in C^2\left([0,x^*]\right), \quad \tilde{\phi}_2 \in C^2\left([x^*,\infty)\right), \quad \tilde{\psi}_1 \in C^1\left([0,x^*]\right),$$
$$\tilde{\psi}_2 \in C^1\left([x^*,\infty)\right), \quad \tilde{\mu}_1 \in C^1\left([0,a^{-1}x^*]\right), \quad \tilde{\mu}_2 \in C^1\left([a^{-1}x^*,\infty)\right)$$

be satisfied. The third mixed problem (7)–(9) *has a unique solution v in the sense of Definition 1. This solution is determined by formulas* (13) *and* (19).

P r o o f 1. The existence of the solution, which satisfies Definition 1 and is determined by formulas (13) and (19), is proved earlier. 2. Let us prove the uniqueness by contradiction. Let there exist two solutions v_1 and v_2 to the third mixed problem (7)–(9) in the sense of Definition 1. Then we can show that their difference $V = v_1 - v_2$ belongs to the class $C^2(\bar{Q})$ and satisfies the following mixed problem

$$\begin{cases} \left(\partial_t^2 - a^2\partial_x^2\right)V(t,x) = 0, \quad 0 < t < \infty, \quad 0 < x < \infty, \\ V(0,x) = \partial_t V(0,x) = 0, \quad 0 \le x < \infty, \\ (h - \partial_x)V(t,0) = 0, \quad 0 \le t < \infty. \end{cases}$$

In turn, the solution $V \equiv 0$ of this problem is unique [24, 25] in the class $C^2(\bar{Q})$. It implies $v_1 - v_2 = 0$.

We note that to construct a solution, we could use the fact that the solution to the mixed problem (7)–(9) can be represented as solutions to the coupled Cauchy, Goursat, and Picard problems. In this approach, the uniqueness follows by design since the solution to these coupled problems is determined in a unique way, e. g., see [22] for the Cauchy problem, [26, 27] for the Goursat problem, and [25] for the Picard problem.

R e m a r k 2. We can easily show that the solution to the problem (7)–(9) in the sense of Definition 1 is a mild solution to the problem (7)–(9) in the sense similar to [28, Definition 1].

3. Main problem. Now we return to the problems (1)–(3) and (4)–(6). The definition of the solution to the (1)–(3) problem was given in the paper [18]. For the convenience of the reader, we present it here in a slightly modified form.

D e f i n i t i o n 2. *Let the matching condition*

$$f_1(0,0) - \mu_0 + c^2\varphi(0) + a_1^2 D^2\varphi(0) + b^2 D\varphi(0) = 0 \tag{20}$$

be satisfied. Then a function u_1 belonging to the class $C(\overline{Q_1}) \cap C^2(Q_-) \cap C^2(Q_+)$, where

$$Q_- = \{(t,x) \mid 0 \le t \le T, x \ge 0, x - a_1 t > 0\},$$

$$Q_+ = \{(t,x) \mid 0 \le t \le T, x \ge 0, x - a_1 t < 0\},$$

is called a classical solution to the problem (1)–(3) *if it satisfies Eq.* (1) *on the sets Q_- and Q_+, the initial conditions* (2) *and the boundary condition* (3) *on the open half-line* $(0,\infty)$*, and the conjugation conditions*

$$[(u_1)^+ - (u_1)^-](t, a_1 t) = 0,$$

$$[(\partial_x u_1)^+ - (\partial_x u_1)^-](t, a_1 t) = \frac{\left(\frac{\psi_1}{2} - C^{(1)} - \frac{\psi_2(0_+)}{2}\right)}{a_1}, \tag{21}$$

where $C^{(1)}$ is an arbitrary real preset constant.

We derive the necessary and sufficient conditions (20) and (21) by passing to the limit. The paper [18] proves that condition (20) is necessary and sufficient, i. e., the quantity μ_0 is uniquely determined, and, otherwise, it is not possible to introduce a definition of the solution. We have already constructed a solution satisfying Definition 2 in the paper [18], and it can be written out in the explicit analytical form

$$u_1(t,x) = w_1(t,x) + \frac{\varphi(x + a_1 t) + \varphi(x - a_1 t)}{2} + \frac{1}{2a_1}\int_{x - a_1 t}^{x + a_1 t} \psi(\xi)d\xi, \quad 0 \le t \le T, \quad x \ge 0, \quad x - a_1 t \ge 0,$$

$$u_2(t,x) = w_1(t,x) + \frac{\varphi(x + a_1 t)}{2} + \frac{1}{2a_1}\int_0^{x + a_1 t} \psi(\xi)d\xi + \frac{\exp\left(\frac{b^2(x - a_1 t)}{2a_1^2}\right)}{2\sqrt{b^4 - 4a_1^2 c^2}} \times$$

$$\times\left(\varphi(0)\cosh\left(\frac{\sqrt{b^4-4a_1^2c^2}\,(x-a_1t)}{2a_1^2}\right)\sqrt{b^4-4a_1^2c^2}+\right.$$

$$+\sinh\left(\frac{\sqrt{b^4-4a_1^2c^2}\,(x-a_1t)}{2a_1^2}\right)\left(2a_1\left(a_1\varphi'(0)+2C^{(1)}-\psi_1\right)-b^2\varphi(0)\right)\Bigg)+$$

$$+\int_0^{x-a_1t}\frac{\exp\left(\dfrac{b^2(x-a_1t-\eta)}{2a_1^2}\right)}{a_1\sqrt{b^4-4a_1^2c^2}}\sinh\left(\frac{\sqrt{b^4-4a_1^2c^2}\,(a_1t+\eta-x)}{2a_1^2}\right)\times$$

$$\times\left(c^2\int_0^{-\eta}\psi(\xi)d\xi-b^2\psi(-\eta)+2a_1c^2w_1\left(-\frac{\eta}{a_1},0\right)+a_1c^2\varphi(-\eta)-\right.$$

$$-2a_1\mu\left(-\frac{\eta}{a_1}\right)-a_1b^2\varphi'(-\eta)+a_1^2\psi_2'(-\eta)+a_1^3\varphi''(-\eta)-2a_1b^2\partial_xw_1\left(-\frac{\eta}{a_1},0\right)+$$

$$\left.+2a_1\partial_t^2w_1\left(-\frac{\eta}{a_1},0\right)\right)d\eta,\quad t\ge T,\quad x\ge 0,\quad x-a_1t\le 0, \tag{22}$$

where

$$w_1(t,x)=\frac{1}{2a_1}\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_1(t-\tau)}f\left(\tau,|\xi|\right)d\xi,\quad t\ge 0,\quad x\ge 0.$$

T h e o r e m 2. *Let the smoothness conditions*

$$f_1\in C^1\left([0,T]\times[0,\infty)\right),\quad \varphi\in C^2\left([0,\infty)\right),\quad \psi_2\in C^1\left([0,\infty)\right),\quad \mu_1\in C^1\left([0,T]\right)$$

be satisfied. The mixed problem (1)–(3) *has a unique solution* u_1 *in the sense of Definition 2. This solution is determined by the formula* (22).

The p r o o f of Theorem 2 is presented in the paper [18].

To solve the problem (4)–(6), we can make a change of the variable $t = t - T$, i. e. look for a solution in the form $u_2(t,x)=v(t+T,x)$. In this case, the function v satisfies the problem

$$\begin{cases}\left(\partial_t^2-a_2^2\partial_x^2\right)v(t,x)=f_2(t-T,x), & 0<t<\infty,\quad 0<x<\infty,\\ v(0,x)=u_1(T,x),\quad \partial_tv(0,x)=\partial_tu_1(T,x), & 0\le x<\infty,\\ (h-\partial_x)v(t,0)=\mu_2(t-T), & 0\le t<\infty.\end{cases} \tag{23}$$

Note that the following conditions are met: 1) the function $\phi_x : x\to u_1(T,x)$ is continuous on the set [0,∞), twice continuously differentiable set $[0,a_1T)\cup(a_1T,\infty)$, and the quantities $D_-^i[\phi_x]$, $D_+^i[\phi_x]$, i = 1,2, where D_-^i is the operator of the i^{th} order left derivative and D_+^i is the operator of the i^{th} order right derivative, exist and take finite values everywhere; 2) the function $\psi_x : x\to\partial_tu_1(T,x)$ is continuous on the set [0,∞), continuously differentiable on the set $[0,a_1T)\cup(a_1T,\infty)$, and the quantities $D_-[\psi_x]$ and $D_+[\psi_x]$ exist and take finite values everywhere; 3) the function $t\to\mu_2(t-T)$ is once continuously differentiable on the set [0,∞) if $\mu_2\in C^1\left([T,\infty)\right)$; 4) the function $t\to f_2(t-T,x)$ is once continuously differentiable on the set $[0,\infty)\times[0,\infty)$ if $f_2\in C^1\left([T,\infty)\times[0,\infty)\right)$. In this case, we can consider the problem (23) as solved, and, consequently, the problem (4)–(6) too.

Let us formulate two equivalent definitions of the solution to the problem (4.4)–(4.6).

D e f i n i t i o n 3. *A function* $u_2:[T,\infty)\times[0,\infty)\mapsto\mathbb{R}$ *is called a classical solution to the problem* (4)–(6) *if the following conditions are fulfilled: 1) the function* u_2 *is twice continuously differentiable and satisfies the equation* (4) *everywhere, except the characteristics* $x-a_2t=a_2T$, $x-a_2t=\pm a_1T-a_2T$ *and* $x+a_2t=a_1T+a_2T$; *2) the first initial condition* $u_2(t=T,x)=u_1(T,x)$ *is satisfied on the entire half-line* $x\geq 0$; *3) the second initial condition* $\partial_t u_2(t=T,x)=\partial_t u_1(T,x)$ *is satisfied on the set* $[0,a_1T)\cup(a_1T,\infty)$; *4) the boundary condition* (5) *holds on the set* $\left[T,a_2^{-1}T(a_1+a_2)\right)\cup\left(a_2^{-1}T(a_1+a_2),\infty\right)$; *5) the function* u_2 *satisfies the following conjugation conditions:*

$$[(u_2)^+-(u_2)^-](t,a_2t\pm a_1T-a_2T)=0,$$

$$[(u_2)^+-(u_2)^-]\big(t,a_2(t+T)\big)=0,$$

$$[(u_2)^+-(u_2)^-](t,a_1T+a_2T-a_2t)=0$$

on the characteristics $x-a_2t=a_2T$, $x-a_2t=\pm a_1T-a_2T$ *and* $x+a_2t=a_1T+a_2T$.

D e f i n i t i o n 4. *A function* u_2 *is called a classical solution to the problem* (4)–(6) *if the function*

$$v:(t,x)\ni[0,\infty)\times[0,\infty)\mapsto u_2(t-T,x)\in\mathbb{R}$$

is a solution to the problem (23) *in the sense of Definition 1.*

Let us formulate the existence and uniqueness theorem for the problem (4)–(6).

T h e o r e m 3. *Let the smoothness conditions*

$$f_1\in C^1\big([0,T]\times[0,\infty)\big),\quad f_2\in C^1\big([T,\infty)\times[0,\infty)\big),\quad \varphi\in C^2\big([0,\infty)\big),$$

$$\psi_2\in C^1\big([0,\infty)\big),\quad \mu_1\in C^1([0,T]),\quad \mu_2\in C^1\big([T,\infty)\big)$$

be satisfied. The third mixed problem (4)–(6) *has a unique solution* u_1 *in the sense of Definition 3 (or Definition 4). This solution is determined by an expression* $u_2(t,x)=v(t-T,x)$, *where the function* v *is determined by the formula* (22)*, in which the number* $a_1=a_2$ *and functions* $\tilde{\phi}:x\to u_1(T,x)$, $\tilde{\psi}:x\to u_1(T,x)$, $\tilde{\mu}:t\to\mu_2(t-T)$.

Thus, according to Theorems 2 and 3, we have constructed the solutions to the problems (1)–(3) and (4)–(6).

4. Duration of the collision. Earlier in the present paper, we assumed that the duration of the collision T is a known quantity. However, we can find it within the Saint-Venant's (vibrational) impact theory. To demonstrate this, consider the simplest case of the problems (1)–(3) and (4)–(6): 1) we assume that at the initial moment of time $t=0$, the rod is at rest, i. e. $\varphi=\psi_2\equiv 0$; 2) we assume that the external forces are absent, i. e. $f_1\equiv f_2\equiv 0$ and $\mu_2\equiv\mu_1\equiv 0$; 3) $a=a_1=a_2$, since the properties of the material of the rod do not change upon impact, and, from physical considerations, $b^2>0$, $c^2\leq 0$. The quantity μ_0 is calculated as $\mu_0=f_1(0,0)+c^2\varphi(0)+a_1^2D^2\varphi(0)+b^2D\varphi(0)=0$. In this case, the solution u_1 to the problem (1)–(3), which satisfies the physically correct matching condition [29, § 2.10]

$$[(\sigma)^+-(\sigma)^-](t,x=at)=v\rho c_u,$$

where $\sigma=E\partial_x u$ is the stress, ρ is the density of the rod material, $c_u=\sqrt{\rho^{-1}E}$ is the propagation speed of deformations along the rod, E is Young's modulus, v is the speed of the body colliding with the rod, is found by the formula

$$u_1(t,x)=\frac{2a\psi_1\exp\left(\dfrac{b^2(x-at)}{2a^2}\right)}{\sqrt{b^4-4a^2c^2}}\theta\left(t-\frac{x}{a}\right)\sinh\left(\frac{(at-x)\sqrt{b^4-4a^2c^2}}{2a^2}\right).\qquad(24)$$

In this case, the contact duration T is determined as the minimum non-negative root of the equation [30, p. 49]

$$\partial_x u_1(T,0)=0. \tag{25}$$

In the case $c \neq 0$ and $b^4-4a^2c^2 \neq 0$, Eq. (25) has the root

$$T=\frac{a}{\sqrt{b^4-4a^2c^2}}\ln\left(\frac{b^4-2a^2c^2+b^2\sqrt{b^4-4a^2c^2}}{2a^2c^2}\right),$$

which is a positive real number. If $c \neq 0$ and $b^4-4a^2c^2=0$, then $\partial_x u_1 \equiv 0$, and therefore $T = 0$, i. e. in this model, the impacting body separates from the rod immediately. If $c = 0$, then there is no linear elastic element at the end $x = 0$ of the rod, i. e. the end $x = 0$ of the rod is free, and Eq. (25) takes the form

$$\frac{b^2}{a^2}\exp\left(\frac{b^2T}{a}\right)=0$$

and has no real roots.

Thus, we have demonstrated a qualitative difference in the impact process in a rod with an end with a linear elastic element and in a rod with a free end.

Of course, within the framework of this longitudinal impact model, we can also pose optimal boundary control problems, where for the given initial state φ and ψ, it is necessary to determine the external force μ_1 so that the load separates from the rod at the given moment of time. However, these problems are beyond the scope of this paper. For similar optimal boundary control problems, we refer the reader to [31–33].

Conclusions. In the present paper, we have studied two coupled mixed problems for the wave equation in a quarter plane from the theory of longitudinal impact. We have formulated the matching conditions under which there is a classical solution to the problems in the case of sufficient smoothness of the given functions. We have constructed a classical solution to two coupled mixed problems and showed the dependence of its smoothness on the given functions. We have proved the uniqueness of the classical solutions. We have analyzed the qualitative difference between the free and the elastically fixed end of the rod. We have obtained an estimate of the duration of a collision.

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