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MASSLESS SPIN 2 FIELD IN 50-COMPONENT APPROACH: EXACT SOLUTIONS WITH CYLINDRICAL SYMMETRY, ELIMINATING THE GUAGE DEGREES OF FREEDOM

Abstract. We begin with some known results of the 50-component theory for a spin-2 field described in cylindrical coordinates. This theory is based on the use of a 2nd-rank symmetric tensor and a 3rd-rank tensor symmetric in two indices. In the massive case, this theory describes a spin-2 particle with an anomalous magnetic moment. According to the Fedorov – Gronskiy method, which is based on projective operators, all 50 functions involved in the description of the spin-2 field for the case of the free particle can be expressed in terms of only 7 different functions constructed from Bessel functions. This leads to a homogeneous system of linear algebraic equations for 50 numerical parameters. We have found 6 independent solutions to these equations. Additionally, we have obtained explicit expressions for 4 guage solutions defined in accordance with the Pauli – Fierz approach. These solutions are exact and correspond to non-physical states that do not affect observable quantities, such as the energy-momentum tensor. Finally, we have constructed two classes of solutions that represent physically observable states.

Keywords: spin 2 massless field, 50-component theory, cylindrical symmetry, Fedorov – Gronskiy method, projective operators, exact solutions, guage symmetry, eliminating the gauge degrees of freedom

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БЕЗМАССОВОЕ ПОЛЕ СО СПИНОМ 2 В 50-КОМПОНЕНТНОМ ПРЕДСТАВЛЕНИИ: ТОЧНЫЕ РЕШЕНИЯ С ЦИЛИНДРИЧЕСКОЙ СИММЕТРИЕЙ, ИСКЛЮЧЕНИЕ КАЛИБРОВОЧНЫХ СТЕПЕНЕЙ СВОБОДЫ

Аннотация. Исходим из известных результатов по 50-компонентной теории для поля со спином 2 в цилиндрических координатах, которая основана на использовании симметричного тензора 2-го ранга и тензора 3-го ранга, симметричного по двум индексам. В массивном случае эта теория описывает частицу со спином 2 с аномальным магнитным моментом. Согласно методу Федорова – Гронского, основанному на проективных операторах, все 50 функций, участвующих в описании поля спина 2 для случая свободной частицы, выражаются через 7 переменных, построенных в терминах функций Бесселя. Для 50 числовых параметров возникает однородная система линейных алгебраических уравнений. В настоящей работе найдены 6 независимых решений данных алгебраических уравнений. Были получены явные выражения для четырех калибровочных решений, определенных в соответствии с подходом Паули – Фирца. Они дают точные решения рассматриваемой системы и относятся к состояниям, которые не дают вклада в физически наблюдаемые величины, такие как тензор энергии-импульса. В итоге построено 2 класса решений, соответствующих физически наблюдаемым состояниям.

Ключевые слова: безмассовое поле со спином 2, 50-компонентная теория, цилиндрическая симметрия, метод Федорова – Гронского, проективные операторы, точные решения, калибровочная симметрия, исключение калибровочных степеней свободы

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Introduction. After the study by Pauli and Fierz [1], the theory of massive and massless fields with spin 2 has always attracted much attention. Most of the studies were performed in the framework of 2nd-order differential equations. However, it is known that many specific difficulties may be avoided if we start with 1st-order systems from the very beginning [2]. Apparently, the first systematic study of the theory of spin 2 field within the first-order formalism was done by Fedorov [3]. This approach requires a field function with 30 independent components. The theory was later re-discovered and improved by Regee [4]. The first-order approach is based, from the very beginning, on the general theory of relativistic wave equations by Gel'fand – Yaglom [2] and Lagrangian formalism.

In the present paper we start with the results obtained in papers [5] concerning the 50-component theory for a spin-2 field with an anomalous magnetic moment, described in cylindrical coordinates. We specify this equation for the massless case as follows:

$$\left[\Gamma^0\frac{\partial}{\partial t}+\Gamma^1\frac{\partial}{\partial r}+\frac{\Gamma^2}{r}\left(\frac{\partial}{\partial\phi}+J^{12}\right)+\Gamma^3\frac{\partial}{\partial z}+P\right]\Psi=0,\qquad \Psi=e^{-i\epsilon t}e^{im\varphi}e^{ikz}\begin{vmatrix}\varphi\\ \varphi_0\\ \varphi_1\\ \varphi_2\\ \varphi_3\end{vmatrix}. \tag{1}$$

Here, P represents the projective operator that separates the constituents of the 3-rank tensor in the complete field function. In the 50-component wave function Ψ we distinguish 10-dimensional components φ, φ_a related to the symmetric tensor and a 3-rank tensor symmetric in two indices. For the main matrices Γ^a we use the blocks presentation. So, instead of (1) we have:

$$\left(K_0{}^0\partial_t+K_0{}^1\partial_r+K_0{}^2\frac{\partial_\varphi}{r}+K_0{}^3\partial_z\right)\varphi_0+\left(K_1{}^0\partial_t+K_1{}^1\partial_r+K_1{}^2\frac{\partial_\varphi}{r}+K_1{}^3\partial_z\right)\varphi_1+$$
$$+\left(K_2{}^0\partial_t+K_2{}^1\partial_r+K_2{}^2\frac{\partial_\varphi}{r}+K_2{}^3\partial_z\right)\varphi_2+\left(K_3{}^0\partial_t+K_3{}^1\partial_r+K_3{}^2\frac{\partial_\varphi}{r}+K_3{}^3\partial_z\right)\varphi_3+$$
$$+\frac{1}{r}\left(K_0{}^2J_1^{12}\varphi_0+\left(K_1{}^2J_1^{12}+K_2{}^2\right)\varphi_1+\left(K_2{}^2J_1^{12}-K_1{}^2\right)\varphi_2+K_3{}^2J_1^{12}\varphi_3\right)=0; \tag{2}$$

$$\begin{aligned}
&\left(L_0{}^0\partial_t+L_0{}^1\partial_r+L_0{}^2\frac{\partial_\varphi}{r}+L_0{}^3\partial_z\right)\varphi+\frac{1}{r}L_0{}^2J_1^{12}\varphi+\varphi_0=0,\\
&\left(L_1{}^0\partial_t+L_1{}^1\partial_r+L_1{}^2\frac{\partial_\varphi}{r}+L_1{}^3\partial_z\right)\varphi+\frac{1}{r}L_1{}^2J_1^{12}\varphi+\varphi_1=0,\\
&\left(L_2{}^0\partial_t+L_2{}^1\partial_r+L_2{}^2\frac{\partial_\varphi}{r}+L_2{}^3\partial_z\right)\varphi+\frac{1}{r}L_2{}^2J_1^{12}\varphi+\varphi_2=0,\\
&\left(L_3{}^0\partial_t+L_3{}^1\partial_r+L_3{}^2\frac{\partial_\varphi}{r}+L_3{}^3\partial_z\right)\varphi+\frac{1}{r}L_3{}^2J_1^{12}\varphi+\varphi_3=0.
\end{aligned} \tag{3}$$

In [5] we solved this equation in the presence of the an external magnetic field. We used the method by Fedorov – Gronskiy [6] that is based on projective operators. This approach allows us to express all 50 functions of the spin-2 field in terms of only 7 different functions constructed in terms of Bessel functions. We derived an algebraic system for the 50 numerical parameters that determine the structure of the 50-component field function. After eliminating 40 components, we obtained an algebraic system of 10 equations, that had 5 independent solutions. In the present paper we specify the previous results to the massless case. First, we have found 6 independent solutions of similar algebraic equations specified for the massless case. Then, we have obtained explicit expressions for 4 guage solutions defined in accordance with the Pauli – Fierz general approach [1]. These solutions provide us with exact solutions of the system under consideration, and relate to states which do not contribute to physically observable quantities like the energy-momentum tensor. Finally, we have constructed two classes of solutions which correspond to physically observable states.

Free spin 2 field. After eliminating 40 variables related to the 3-rank tensor (see in [5, 6]), we obtain the following system of homogeneous algebraic equations for 10 variables, which relate to the 10-component symmetric tensor (in this point we still follow the massive case, $M \neq 0$):

$$2ic_3k\sqrt{y}+c_2\left(y-\frac{y}{\sqrt{3}}\right)+2id_1\sqrt{y}\epsilon+f_1(-k^2+M^2+\epsilon^2)-\frac{1}{6}(3+\sqrt{3})f_0y+\frac{1}{6}(3+\sqrt{3})f_2y=0,$$

$$\frac{1}{6}c_2\left((3+2\sqrt{3})k^2+2\sqrt{3}y+3\epsilon^2\right)-\frac{1}{3}i(3+\sqrt{3})c_1k\sqrt{y}+\frac{1}{3}i(3+\sqrt{3})c_3k\sqrt{y}+\frac{1}{3}(3+\sqrt{3})d_2k\epsilon+$$
$$+\frac{1}{3}i(\sqrt{3}-3)d_1\sqrt{y}\epsilon-\frac{1}{3}i(\sqrt{3}-3)d_3\sqrt{y}\epsilon+\frac{1}{4}f_2(k^2+4M^2+6y+3\epsilon^2)+\frac{1}{12}f_0\left((3+2\sqrt{3})k^2+6y+(2\sqrt{3}-3)\epsilon^2\right)-$$
$$-\frac{1}{6}(\sqrt{3}-3)f_1y-\frac{1}{6}(\sqrt{3}-3)f_3y=0,$$

$$-2ic_1k\sqrt{y}+c_2\left(y-\frac{y}{\sqrt{3}}\right)-2id_3\sqrt{y}\epsilon+f_3(-k^2+M^2+\epsilon^2)-\frac{1}{6}(3+\sqrt{3})f_0y+\frac{1}{6}(3+\sqrt{3})f_2y=0,$$

$$-\frac{ic_2k\sqrt{y}}{\sqrt{3}}+c_1(M^2+y+\epsilon^2)+c_3y+d_3k\epsilon-id_2\sqrt{y}\epsilon-\frac{1}{6}i(3+\sqrt{3})f_0k\sqrt{y}+\frac{1}{6}i(\sqrt{3}-3)f_2k\sqrt{y}-if_3k\sqrt{y}=0,$$

$$c_2\left(-\frac{k^2}{2}+M^2+y+\frac{\epsilon^2}{2}\right)+\frac{ic_1k\sqrt{y}}{\sqrt{3}}-\frac{ic_3k\sqrt{y}}{\sqrt{3}}-\frac{1}{3}(\sqrt{3}-3)d_2k\epsilon-\frac{id_1\sqrt{y}\epsilon}{\sqrt{3}}+\frac{id_3\sqrt{y}\epsilon}{\sqrt{3}}+$$
$$+\frac{1}{12}f_0\left(3k^2+2\sqrt{3}y+(3-2\sqrt{3})\epsilon^2\right)+\frac{1}{12}f_2\left((3-2\sqrt{3})k^2-2\sqrt{3}y+3\epsilon^2\right)+\frac{1}{6}(3+\sqrt{3})f_1y+\frac{1}{6}(3+\sqrt{3})f_3y=0,$$

$$\frac{ic_2k\sqrt{y}}{\sqrt{3}}+c_3(M^2+y+\epsilon^2)+c_1y+d_1k\epsilon+id_2\sqrt{y}\epsilon+\frac{1}{6}i(3+\sqrt{3})f_0k\sqrt{y}+if_1k\sqrt{y}-\frac{1}{6}i(\sqrt{3}-3)f_2k\sqrt{y}=0,$$

$$-c_3k\epsilon-\frac{ic_2\sqrt{y}\epsilon}{\sqrt{3}}+d_1(-k^2+M^2+y)+id_2k\sqrt{y}+d_3y-\frac{1}{6}i(\sqrt{3}-3)f_0\sqrt{y}\epsilon-if_1\sqrt{y}\epsilon+\frac{1}{6}i(3+\sqrt{3})f_2\sqrt{y}\epsilon=0,$$

$$-\frac{1}{3}(3+\sqrt{3})c_2k\epsilon+ic_1\sqrt{y}\epsilon-ic_3\sqrt{y}\epsilon+id_1k\sqrt{y}-id_3k\sqrt{y}+d_2(M^2+2y)-\frac{1}{6}(\sqrt{3}-3)f_0k\epsilon+\frac{1}{6}(\sqrt{3}-3)f_2k\epsilon=0,$$

$$c_1(-k)\epsilon+\frac{ic_2\sqrt{y}\epsilon}{\sqrt{3}}+d_3(-k^2+M^2+y)-id_2k\sqrt{y}+d_1y+\frac{1}{6}i(\sqrt{3}-3)f_0\sqrt{y}\epsilon-\frac{1}{6}i(3+\sqrt{3})f_2\sqrt{y}\epsilon+if_3\sqrt{y}\epsilon=0,$$

$$\frac{1}{6}c_2\left(3k^2-2\sqrt{3}y+(3+2\sqrt{3})\epsilon^2\right)+\frac{1}{3}i(\sqrt{3}-3)c_1k\sqrt{y}-\frac{1}{3}i(\sqrt{3}-3)c_3k\sqrt{y}-\frac{1}{3}(3+\sqrt{3})d_2k\epsilon-$$
$$-\frac{1}{3}i(3+\sqrt{3})d_1\sqrt{y}\epsilon+\frac{1}{3}i(3+\sqrt{3})d_3\sqrt{y}\epsilon+\frac{1}{4}f_0(-3k^2+4M^2+6y-\epsilon^2)+$$
$$+\frac{1}{12}f_2\left((3-2\sqrt{3})k^2+6y-(3+2\sqrt{3})\epsilon^2\right)+\frac{1}{6}(\sqrt{3}-3)f_1y+\frac{1}{6}(\sqrt{3}-3)f_3y=0. \quad (4)$$

This system can be presented in matrix form $AF = 0$, where $F=\{f_1,f_2,f_3,c_1,c_2,c_3,d_1,d_2,d_3,f_0\}$ determines the structure of the 10 components of the symmetric tensor. From the vanishing of the determinant, we derive the equation $M^{10}(\epsilon^2 - k^2 + M^2 + 2y)^5 = 0$ (the quantity y was introduced as an arbitrary parameter within the Fedorov – Gronskiy method [6]). It becomes obvious that the sign before the parameter M^2 must be reversed, $M = i\mu,\ M^2 = -\mu^2.$ So we get

$$\text{I} \quad \sqrt{y} = -i\frac{\sqrt{\epsilon^2 - \mu^2 - k^2}}{\sqrt{2}}, \qquad \text{II} \quad \sqrt{y} = +i\frac{\sqrt{\epsilon^2 - \mu^2 - k^2}}{\sqrt{2}}. \tag{5}$$

For definiteness, we will follow the possibility I. With this in mind, the rank of the matrix turns out to be 5. We delete rows 1, 2, 4, 5, 8 and move the columns corresponding to the variables f_1, c_1, c_2, d_3, f_0 to the right-hand side. As a result, we obtain a system of 5 equations with 5 free parameters, the solutions of which can be readily found.

Massless case. The rank of the relevant matrix in the massless case turns out to be equal to 4. We delete rows 1, 2, 3, 4, 5, 8 and move the columns corresponding to the variables $f_1, f_2, c_1, c_2, d_1, d_2$ to the right-hand side. We can readily find the solutions of this system. The 6 independent solutions can be presented in 10-dimensional form as follows:

$$\Psi_1 = f_1 \left| \begin{array}{c} 1 \\ 0 \\ \dfrac{k^4 + (11 - 3\sqrt{3})k^2\epsilon^2 + (6+\sqrt{3})\epsilon^4}{(3k^2 - \epsilon^2)\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \\ 0 \\ 0 \\ -\dfrac{\sqrt{2}k\sqrt{\epsilon^2 - k^2}}{3k^2 - \epsilon^2} \\ 0 \\ 0 \\ -\dfrac{\sqrt{2}\epsilon\sqrt{\epsilon^2 - k^2}\left((5 - 2\sqrt{3})k^2 + (4+\sqrt{3})\epsilon^2\right)}{(\epsilon^2 - 3k^2)\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \\ \dfrac{2(k^2 - \epsilon^2)\left((\sqrt{3}-3)k^2 - (3+\sqrt{3})\epsilon^2\right)}{(3k^2 - \epsilon^2)\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \end{array} \right|, \quad \Psi_2 = f_2 \left| \begin{array}{c} 0 \\ 1 \\ \dfrac{2\left(k^6 + (8 - 3\sqrt{3})k^4\epsilon^2 + (1 - 2\sqrt{3})k^2\epsilon^4 + (2+\sqrt{3})\epsilon^6\right)}{(k^2 - \epsilon^2)(3k^2 - \epsilon^2)\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \\ 0 \\ 0 \\ \dfrac{\sqrt{2}k\sqrt{\epsilon^2 - k^2}}{3k^2 - \epsilon^2} \\ 0 \\ 0 \\ \dfrac{\sqrt{2}\epsilon\left((7 - 4\sqrt{3})k^4 - (\sqrt{3}-3)k^2\epsilon^2 + (2+\sqrt{3})\epsilon^4\right)}{(\epsilon^2 - 3k^2)\sqrt{\epsilon^2 - k^2}\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \\ \dfrac{(4\sqrt{3} - 9)k^4 - (7+\sqrt{3})k^2\epsilon^2 - (2+\sqrt{3})\epsilon^4}{(3k^2 - \epsilon^2)\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \end{array} \right|,$$

$$\Psi_3 = c_1 \left| \begin{array}{c} 0 \\ 0 \\ \dfrac{4\sqrt{2}k(k^4 + 4k^2\epsilon^2 + \epsilon^4)}{(3k^2 - \epsilon^2)\sqrt{\epsilon^2 - k^2}\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \\ 1 \\ 0 \\ \dfrac{k^2 - \epsilon^2}{3k^2 - \epsilon^2} \\ 0 \\ 0 \\ \dfrac{2k\epsilon\left((5 - 2\sqrt{3})k^2 + (4+\sqrt{3})\epsilon^2\right)}{(3k^2 - \epsilon^2)\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \\ -\dfrac{2\sqrt{2}k\sqrt{\epsilon^2 - k^2}\left((\sqrt{3}-3)k^2 - (3+\sqrt{3})\epsilon^2\right)}{(3k^2 - \epsilon^2)\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \end{array} \right|, \quad \Psi_4 = c_2 \left| \begin{array}{c} 0 \\ 0 \\ \dfrac{2\left(k^6 + (2+3\sqrt{3})k^4\epsilon^2 + (9+2\sqrt{3})k^2\epsilon^4 - \sqrt{3}\epsilon^6\right)}{(k^2 - \epsilon^2)(3k^2 - \epsilon^2)\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \\ 0 \\ 1 \\ \dfrac{\sqrt{2}k\sqrt{\epsilon^2 - k^2}}{3k^2 - \epsilon^2} \\ 0 \\ 0 \\ \dfrac{\sqrt{2}\epsilon\left((1+2\sqrt{3})k^4 + (11+3\sqrt{3})k^2\epsilon^2 - \sqrt{3}\epsilon^4\right)}{(\epsilon^2 - 3k^2)\sqrt{\epsilon^2 - k^2}\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \\ -\dfrac{2\left(\sqrt{3}k^4 + (5+\sqrt{3})k^2\epsilon^2 + \epsilon^4\right)}{(3k^2 - \epsilon^2)\left(k^2 + (2+\sqrt{3})\epsilon^2\right)} \end{array} \right|,$$

$$\Psi_5 = d_1 \left| \begin{array}{c} 0 \\ 0 \\ \dfrac{2\left(k^6+(2+3\sqrt{3})k^4\epsilon^2+(9+2\sqrt{3})k^2\epsilon^4-\sqrt{3}\epsilon^6\right)}{(k^2-\epsilon^2)(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} \\ 0 \\ 1 \\ \dfrac{\sqrt{2}k\sqrt{\epsilon^2-k^2}}{3k^2-\epsilon^2} \\ 0 \\ 0 \\ \dfrac{\sqrt{2}\epsilon\left((1+2\sqrt{3})k^4+(11+3\sqrt{3})k^2\epsilon^2-\sqrt{3}\epsilon^4\right)}{(\epsilon^2-3k^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)} \\ -\dfrac{2\left(\sqrt{3}k^4+(5+\sqrt{3})k^2\epsilon^2+\epsilon^4\right)}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} \end{array} \right|, \quad \Psi_6 = d_2 \left| \begin{array}{c} 0 \\ 0 \\ -\dfrac{4k\epsilon\left(k^4+3(\sqrt{3}-1)k^2\epsilon^2-(4+\sqrt{3})\epsilon^4\right)}{(k^2-\epsilon^2)(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} \\ 0 \\ 0 \\ -\dfrac{\sqrt{2}\epsilon\sqrt{\epsilon^2-k^2}}{\epsilon^2-3k^2} \\ 0 \\ 1 \\ \dfrac{\sqrt{2}k\left(3k^4+(7\sqrt{3}-5)k^2\epsilon^2-(10+3\sqrt{3})\epsilon^4\right)}{(3k^2-\epsilon^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)} \\ \dfrac{4(\sqrt{3}-3)k^3\epsilon-4(3+\sqrt{3})k\epsilon^3}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} \end{array} \right|.$$

Keeping in mind the expressions of 10 variables in terms of basic 5 functions (see [5, 6]), we can rewrite the previous formulas differently:

$$\Psi_1 = f_1 \left| \begin{array}{c} L_4 J_{m-2} \\ 0 \\ \dfrac{k^4+(11-3\sqrt{3})k^2\epsilon^2+(6+\sqrt{3})\epsilon^4}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_5 J_{m+2} \\ 0 \\ 0 \\ -\dfrac{\sqrt{2}k\sqrt{\epsilon^2-k^2}}{3k^2-\epsilon^2} L_2 J_{m-1} \\ 0 \\ 0 \\ -\dfrac{\sqrt{2}\epsilon\sqrt{\epsilon^2-k^2}\left((5-2\sqrt{3})k^2+(4+\sqrt{3})\epsilon^2\right)}{(\epsilon^2-3k^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_3 J_{m+1} \\ \dfrac{2(k^2-\epsilon^2)\left((\sqrt{3}-3)k^2-(3+\sqrt{3})\epsilon^2\right)}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_1 J_m \end{array} \right|,$$

$$\Psi_2 = f_2 \left| \begin{matrix} 0 \\ L_1 J_m \\ \dfrac{2\left(k^6+(8-3\sqrt{3})k^4\epsilon^2+(1-2\sqrt{3})k^2\epsilon^4+(2+\sqrt{3})\epsilon^6\right)}{(k^2-\epsilon^2)(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_5 J_{m+2} \\ 0 \\ 0 \\ \dfrac{\sqrt{2}k\sqrt{\epsilon^2-k^2}}{3k^2-\epsilon^2} L_2 J_{m-1} \\ 0 \\ 0 \\ \dfrac{\sqrt{2}\epsilon\left((7-4\sqrt{3})k^4-(\sqrt{3}-3)k^2\epsilon^2+(2+\sqrt{3})\epsilon^4\right)}{(\epsilon^2-3k^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_3 J_{m+1} \\ \dfrac{(4\sqrt{3}-9)k^4-(7+\sqrt{3})k^2\epsilon^2-(2+\sqrt{3})\epsilon^4}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_1 J_m \end{matrix} \right|,$$

$$\Psi_3 = c_1 \left| \begin{matrix} 0 \\ 0 \\ \dfrac{4\sqrt{2}k(k^4+4k^2\epsilon^2+\epsilon^4)}{(3k^2-\epsilon^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_5 J_{m+2} \\ L_3 J_{m+1} \\ 0 \\ \dfrac{k^2-\epsilon^2}{3k^2-\epsilon^2} L_2 J_{m-1} \\ 0 \\ 0 \\ \dfrac{2k\epsilon\left((5-2\sqrt{3})k^2+(4+\sqrt{3})\epsilon^2\right)}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_3 J_{m+1} \\ -\dfrac{2\sqrt{2}k\sqrt{\epsilon^2-k^2}\left((\sqrt{3}-3)k^2-(3+\sqrt{3})\epsilon^2\right)}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_1 J_m \end{matrix} \right|, \Psi_4 = c_2 \left| \begin{matrix} 0 \\ 0 \\ \dfrac{2\left(k^6+(2+3\sqrt{3})k^4\epsilon^2+(9+2\sqrt{3})k^2\epsilon^4-\sqrt{3}\epsilon^6\right)}{(k^2-\epsilon^2)(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_5 J_{m+2} \\ 0 \\ L_1 J_m \\ \dfrac{\sqrt{2}k\sqrt{\epsilon^2-k^2}}{3k^2-\epsilon^2} L_2 J_{m-1} \\ 0 \\ 0 \\ \dfrac{\sqrt{2}\epsilon\left((1+2\sqrt{3})k^4+(11+3\sqrt{3})k^2\epsilon^2-\sqrt{3}\epsilon^4\right)}{(\epsilon^2-3k^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_3 J_{m+1} \\ -\dfrac{2\left(\sqrt{3}k^4+(5+\sqrt{3})k^2\epsilon^2+\epsilon^4\right)}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_1 J_m \end{matrix} \right|,$$

$$\Psi_5 = d_1 \left| \begin{matrix} 0 \\ 0 \\ -\dfrac{4\sqrt{2}\epsilon(k^4+4k^2\epsilon^2+\epsilon^4)}{(\epsilon^2-3k^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_5 J_{m+2} \\ 0 \\ 0 \\ -\dfrac{2k\epsilon}{3k^2-\epsilon^2} L_2 J_{m-1} \\ L_2 J_{m-1} \\ 0 \\ \dfrac{3k^4-(\sqrt{3}-15)k^2\epsilon^2+(6+\sqrt{3})\epsilon^4}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_3 J_{m+1} \\ -\dfrac{2\sqrt{2}\epsilon\sqrt{\epsilon^2-k^2}\left((3+\sqrt{3})\epsilon^2-(\sqrt{3}-3)k^2\right)}{(\epsilon^2-3k^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_1 J_m \end{matrix} \right|, \Psi_6 = d_2 \left| \begin{matrix} 0 \\ 0 \\ -\dfrac{4k\epsilon\left(k^4+3(\sqrt{3}-1)k^2\epsilon^2-(4+\sqrt{3})\epsilon^4\right)}{(k^2-\epsilon^2)(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_5 J_{m+2} \\ 0 \\ 0 \\ -\dfrac{\sqrt{2}\epsilon\sqrt{\epsilon^2-k^2}}{\epsilon^2-3k^2} L_2 J_{m-1} \\ 0 \\ L_1 J_m \\ \dfrac{\sqrt{2}k\left(3k^4+(7\sqrt{3}-5)k^2\epsilon^2-(10+3\sqrt{3})\epsilon^4\right)}{(3k^2-\epsilon^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_3 J_{m+1} \\ \dfrac{4(\sqrt{3}-3)k^3\epsilon-4(3+\sqrt{3})k\epsilon^3}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} L_1 J_m \end{matrix} \right|.$$

Parameters $L_1,...,L_5$ are related by linear constraints. Thus, we may present 6 independent solutions in a more simple form

$$\Psi_1 = f_1 \left| \begin{array}{c} -J_{m-2} \\ 0 \\ -\dfrac{k^4+(11-3\sqrt{3})k^2\epsilon^2+(6+\sqrt{3})\epsilon^4}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} J_{m+2} \\ 0 \\ 0 \\ -i\dfrac{\sqrt{2}k\sqrt{\epsilon^2-k^2}}{3k^2-\epsilon^2} J_{m-1} \\ 0 \\ 0 \\ i\dfrac{\sqrt{2}\epsilon\sqrt{\epsilon^2-k^2}\left((5-2\sqrt{3})k^2+(4+\sqrt{3})\epsilon^2\right)}{(\epsilon^2-3k^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} J_{m+1} \\ \dfrac{2(k^2-\epsilon^2)\left((\sqrt{3}-3)k^2-(3+\sqrt{3})\epsilon^2\right)}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} J_m \end{array} \right|,$$

$$\Psi_2 = f_2 \left| \begin{array}{c} 0 \\ J_m \\ -\dfrac{2\left(k^6+(8-3\sqrt{3})k^4\epsilon^2+(1-2\sqrt{3})k^2\epsilon^4+(2+\sqrt{3})\epsilon^6\right)}{(k^2-\epsilon^2)(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} J_{m+2} \\ 0 \\ 0 \\ i\dfrac{\sqrt{2}k\sqrt{\epsilon^2-k^2}}{3k^2-\epsilon^2} J_{m-1} \\ 0 \\ 0 \\ -i\dfrac{\sqrt{2}\epsilon\left((7-4\sqrt{3})k^4-(\sqrt{3}-3)k^2\epsilon^2+(2+\sqrt{3})\epsilon^4\right)}{(\epsilon^2-3k^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)} J_{m+1} \\ \dfrac{(4\sqrt{3}-9)k^4-(7+\sqrt{3})k^2\epsilon^2-(2+\sqrt{3})\epsilon^4}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)} J_m \end{array} \right|,$$

$$\Psi_3=c_1\left|\begin{array}{c} 0\\ 0\\ -\dfrac{4\sqrt{2}k(k^4+4k^2\epsilon^2+\epsilon^4)}{(3k^2-\epsilon^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_{m+2}\\ -iJ_{m+1}\\ 0\\ i\dfrac{k^2-\epsilon^2}{3k^2-\epsilon^2}J_{m-1}\\ 0\\ 0\\ -i\dfrac{2k\epsilon\left((5-2\sqrt{3})k^2+(4+\sqrt{3})\epsilon^2\right)}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_{m+1}\\ -\dfrac{2\sqrt{2}k\sqrt{\epsilon^2-k^2}\left((\sqrt{3}-3)k^2-(3+\sqrt{3})\epsilon^2\right)}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_m \end{array}\right|,\Psi_4=c_2\left|\begin{array}{c} 0\\ 0\\ -\dfrac{2\left(k^6+(2+3\sqrt{3})k^4\epsilon^2+(9+2\sqrt{3})k^2\epsilon^4-\sqrt{3}\epsilon^6\right)}{(k^2-\epsilon^2)(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_{m+2}\\ 0\\ J_m\\ i\dfrac{\sqrt{2}k\sqrt{\epsilon^2-k^2}}{3k^2-\epsilon^2}J_{m-1}\\ 0\\ 0\\ -i\dfrac{\sqrt{2}\epsilon\left((1+2\sqrt{3})k^4+(11+3\sqrt{3})k^2\epsilon^2-\sqrt{3}\epsilon^4\right)}{(\epsilon^2-3k^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_{m+1}\\ -\dfrac{2(\sqrt{3}k^4+(5+\sqrt{3})k^2\epsilon^2+\epsilon^4)}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_m \end{array}\right|,$$

$$\Psi_5=d_1\left|\begin{array}{c} 0\\ 0\\ \dfrac{4\sqrt{2}\epsilon(k^4+4k^2\epsilon^2+\epsilon^4)}{(\epsilon^2-3k^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_{m+2}\\ 0\\ 0\\ -i\dfrac{2k\epsilon}{3k^2-\epsilon^2}J_{m-1}\\ iJ_{m-1}\\ 0\\ -i\dfrac{3k^4-(\sqrt{3}-15)k^2\epsilon^2+(6+\sqrt{3})\epsilon^4}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_{m+1}\\ -\dfrac{2\sqrt{2}\epsilon\sqrt{\epsilon^2-k^2}\left((3+\sqrt{3})\epsilon^2-(\sqrt{3}-3)k^2\right)}{(\epsilon^2-3k^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_m \end{array}\right|,\Psi_6=d_2\left|\begin{array}{c} 0\\ 0\\ \dfrac{4k\epsilon\left(k^4+3(\sqrt{3}-1)k^2\epsilon^2-(4+\sqrt{3})\epsilon^4\right)}{(k^2-\epsilon^2)(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_{m+2}\\ 0\\ 0\\ -i\dfrac{\sqrt{2}\epsilon\sqrt{\epsilon^2-k^2}}{\epsilon^2-3k^2}J_{m-1}\\ 0\\ J_m\\ -i\dfrac{\sqrt{2}k\left(3k^4+(7\sqrt{3}-5)k^2\epsilon^2-(10+3\sqrt{3})\epsilon^4\right)}{(3k^2-\epsilon^2)\sqrt{\epsilon^2-k^2}\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_{m+1}\\ \dfrac{4(\sqrt{3}-3)k^3\epsilon-4(3+\sqrt{3})k\epsilon^3}{(3k^2-\epsilon^2)\left(k^2+(2+\sqrt{3})\epsilon^2\right)}J_m \end{array}\right|.$$

Guage solutions for massless spin 2 field. As is known, for massless spin 2 field there exist guage solutions [1], that are defined through an arbitrary vector $L_c^C(x), c=0,1,2,3;$ the symbol C designates Cartesian basis:

$$\Phi_{ab}^C(x)=\partial_a L_b^C(x)+\partial_b L_a^C(x)+z\left(-\frac{1}{2}g_{ab}\partial^l L_l^C(x)\right). \qquad (6)$$

This relation in detailed form reads [10]:

$$f_1^C=\Phi_{11}^C=2\partial_r L_1^C+z\frac{1}{2}\left[\partial_t L_0^C-\left(\partial_r+\frac{1}{r}\right)L_1^C-\frac{1}{r}\partial_\phi L_2^C-\partial_z L_3^C\right],$$

$$f_2^C = \Phi_{22}^C = \frac{2}{r}L_1^C + \frac{2}{r}\partial_\phi L_2^C + z\frac{1}{2}\left[\partial_t L_0^C - \left(\partial_r + \frac{1}{r}\right)L_1^C - \frac{1}{r}\partial_\phi L_2^C - \partial_z L_3^C\right],$$

$$f_3^C = \Phi_{33}^C = 2\partial_z L_3^C + z\frac{1}{2}\left[\partial_t L_0^C - \left(\partial_r + \frac{1}{r}\right)L_1^C - \frac{1}{r}\partial_\phi L_2^C - \partial_z L_3^C\right],$$

$$c_1^C = \Phi_{23}^C = \frac{1}{r}\partial_\phi L_3^C + \partial_z L_2^C, \quad c_2^C = \Phi_{31}^C = \partial_z L_1^C + \partial_r L_3^C, \quad c_3^C = \Phi_{12}^C = -\frac{1}{r}L_2^C + \partial_r L_2^C + \frac{1}{r}\partial_\phi L_1^C,$$

$$d_1^C = \Phi_{01}^C = \partial_t L_1^C + \partial_r L_0^C, \quad d_2^C = \Phi_{02}^C = \partial_t L_2^C + \frac{1}{r}\partial_\phi L_0^C, \quad d_3^c = \Phi_{03}^C = \partial_t L_3^C + \partial_z L_0^C;$$

$$f_0^C = \Phi_{00}^C = 2\partial_t L_0^C - z\frac{1}{2}\left[\partial_t L_0^C - \left(\partial_r + \frac{1}{r}\right)L_1^C - \frac{1}{r}\partial_\phi L_2^C - \partial_z L_3^C\right]. \tag{7}$$

Taking into account the structure of the gauge vector [7]:

$$L^C(t,r,\varphi,z) = e^{-i\epsilon t}e^{im\varphi}e^{ikz}\begin{vmatrix} L_0^C \\ L_1^C \\ L_2^C \\ L_3^C \end{vmatrix} = e^{-i\epsilon t}e^{im\varphi}e^{ikz}\begin{vmatrix} \bar{L}_0 \\ \frac{1}{\sqrt{2}}(\bar{L}_3 - \bar{L}_1) \\ \frac{-i}{\sqrt{2}}(\bar{L}_3 + \bar{L}_1) \\ \bar{L}_2 \end{vmatrix}, \tag{8}$$

we find expressions for the components of the corresponding gauge tensor:

$$f_1^C = 2\partial_r L_1^C + z\frac{1}{2}\left[-i\epsilon L_0^C - \left(\partial_r + \frac{1}{r}\right)L_1^C - \frac{1}{r}imL_2^C - ikL_3^C\right],$$

$$f_2^C = \frac{2}{r}L_1^C + \frac{2}{r}imL_2^C + z\frac{1}{2}\left[-i\epsilon L_0^C - \left(\partial_r + \frac{1}{r}\right)L_1^C - \frac{1}{r}imL_2^C - ikL_3^C\right],$$

$$f_3^C = 2ikL_3^C + z\frac{1}{2}\left[-i\epsilon L_0^C - \left(\partial_r + \frac{1}{r}\right)L_1^C - \frac{1}{r}imL_2^C - ikL_3^C\right],$$

$$c_1^C = \frac{1}{r}imL_3^C + ikL_2^C, \quad c_2^C = ikL_1^C + \partial_r L_3^C, \quad c_3^C = -\frac{1}{r}L_2^C + \partial_r L_2^C + \frac{1}{r}imL_1^C,$$

$$d_1^C = -i\epsilon L_1^C + \partial_r L_0^C, \quad d_2^C = -i\epsilon L_2^C + \frac{1}{r}imL_0^C, \quad d_3^C = -i\epsilon L_3^C + ikL_0^C;$$

$$f_0^C = -2i\epsilon L_0^C - z\frac{1}{2}\left[-i\epsilon L_0^C - \left(\partial_r + \frac{1}{r}\right)L_1^C - 1rimL_2^C - ikL_3^C\right]. \tag{9}$$

After transition to cyclic basis (for more detail see in [7]), we obtain (changing to the variable $x = \lambda r,\ \lambda = \sqrt{\epsilon^2 - k^2}$ leads to new presentations for the gauge tensor in cyclic basis):

$$\bar{f}_1 = -\sqrt{2}\lambda\left(\frac{d}{dx} + \frac{m-1}{x}\right)\bar{L}_1,$$

$$\bar{f}_2 = -\frac{1}{2}ik(z-4)\bar{L}_2 - \frac{1}{2}iz\epsilon\bar{L}_0 - \frac{\sqrt{2}}{4}z\lambda\left(\frac{d}{dx} + \frac{m+1}{x}\right)\bar{L}_3 + \frac{\sqrt{2}}{4}z\lambda\left(\frac{d}{dx} - \frac{m-1}{x}\right)\bar{L}_1,$$

$$\bar{f}_3 = \sqrt{2}\lambda\left(\frac{d}{dx} - \frac{m+1}{x}\right)\bar{L}_3, \quad \bar{c}_1 = ik\bar{L}_3 + \frac{1}{\sqrt{2}}\lambda\left(\frac{d}{dx} - \frac{m}{x}\right)\bar{L}_2,$$

$$\bar{c}_2 = \frac{1}{2}ikz\bar{L}_2 + \frac{1}{2}iz\epsilon\bar{L}_0 + \frac{\sqrt{2}}{4}(z-2)\lambda\left(\frac{d}{dx}+\frac{m+1}{x}\right)\bar{L}_3 - \frac{\sqrt{2}}{4}\lambda(z-2)\left(\frac{d}{dx}-\frac{m-1}{x}\right)\bar{L}_1,$$

$$\bar{c}_3 = -\frac{1}{\sqrt{2}}\lambda\left(\frac{d}{dx}+\frac{m}{x}\right)\bar{L}_2 + ik\bar{L}_1, \quad \bar{d}_1 = -\frac{1}{\sqrt{2}}\lambda\left(\frac{d}{dx}+\frac{m}{x}\right)\bar{L}_0 - i\epsilon\bar{L}_1,$$

$$\bar{d}_2 = ik\bar{L}_0 - i\epsilon\bar{L}_2, \quad \bar{d}_3 = \frac{1}{\sqrt{2}}\lambda\left(\frac{d}{dx}-\frac{m}{x}\right)\bar{L}_0 - i\epsilon\bar{L}_3,$$

$$\bar{f}_0 = \frac{1}{2}zik\bar{L}_2 + \frac{1}{2}i(z-4)\epsilon\bar{L}_0 + \frac{z}{2\sqrt{2}}\lambda\left(\frac{d}{dx}+\frac{m+1}{x}\right)\bar{L}_3 - \lambda\frac{z}{2\sqrt{2}}\left(\frac{d}{dx}-\frac{m-1}{x}\right)\bar{L}_1.$$

In [7] four independent solutions in cyclic basis were found for massless spin 1 field:

$$L_0^{(1)} = \frac{\lambda}{\sqrt{2}\epsilon}J_m, \quad L_1^{(1)} = -i\,J_{m-1}, \quad L_2^{(1)} = 0, \quad L_3^{(1)} = 0;$$

$$L_0^{(2)} = -\frac{k}{\epsilon}J_m, \quad L_1^{(2)} = 0, \quad L_2^{(2)} = J_m, \quad L_3^{(2)} = 0;$$

$$L_0^{(3)} = -\frac{\lambda}{\sqrt{2}\epsilon}J_m, \quad L_1^{(3)} = 0, \quad L_2^{(3)} = 0, \quad L_3^{(3)} = +i\,J_{m+1};$$

$$L_0^{(4)} = -iJ_m, \quad L_1^{(4)} = -\frac{\lambda}{\sqrt{2}\epsilon}J_{m-1}, \quad L_2^{(4)} = \frac{ik}{\epsilon}J_m, \quad L_3^{(4)} = -\frac{\lambda}{\sqrt{2}\epsilon}J_{m+1}. \tag{10}$$

Substituting these expressions in the above formulas, we find the corresponding gauge tensors; we write their 10-dimensional form

$$Q = \left|\begin{array}{c} \sqrt{2}i\lambda J_{m-2} \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}}i\lambda J_m \\ kJ_{m-1} \\ -\left(\frac{1}{2\epsilon}\lambda^2+\epsilon\right)J_{m-1} \\ ik\frac{1}{\sqrt{2}\epsilon}\lambda J_m \\ -\frac{1}{2\epsilon}\lambda^2 J_{m+1} \\ -\sqrt{2}i\lambda J_m \end{array}\right|, \quad R = \left|\begin{array}{c} 0 \\ 2ikJ_m \\ 0 \\ -\frac{1}{\sqrt{2}}\lambda J_{m+1} \\ 0 \\ -\frac{1}{\sqrt{2}}\lambda J_{m-1} \\ \frac{1}{\sqrt{2}}\lambda\frac{k}{\epsilon}J_{m-1} \\ -i\left(\frac{k^2}{\epsilon}+\epsilon\right)J_m \\ \frac{1}{\sqrt{2}}\lambda\frac{k}{\epsilon}J_{m+1} \\ 2ikJ_m \end{array}\right|, \quad S = \left|\begin{array}{c} 0 \\ 0 \\ -\sqrt{2}i\lambda J_{m+2} \\ -kJ_{m+1} \\ -\frac{1}{\sqrt{2}}i\lambda J_m \\ 0 \\ \frac{1}{2}\lambda^2\frac{1}{\epsilon}J_{m-1} \\ -ik\frac{1}{\sqrt{2}\epsilon}\lambda J_m \\ \left(\frac{1}{2\epsilon}\lambda^2+\epsilon\right)J_{m+1} \\ \sqrt{2}i\lambda J_m \end{array}\right|, \quad T = \left|\begin{array}{c} \lambda^2\frac{1}{\epsilon}J_{m-2} \\ -\frac{2k^2}{\epsilon}J_m \\ \frac{1}{\epsilon}\lambda^2 J_{m+2} \\ -ik\frac{\sqrt{2}}{\epsilon}\lambda J_{m+1} \\ \frac{\lambda^2}{\epsilon}J_m \\ -ik\frac{\sqrt{2}}{\epsilon}\lambda J_{m-1} \\ \sqrt{2}i\lambda J_{m-1} \\ 2kJ_m \\ \sqrt{2}i\lambda J_{m+1} \\ -2\epsilon J_m \end{array}\right|.$$

It should be noted that the 4 gauge tensors Q, R, S, T do not depend at all on the additional numerical parameter z. This can be understood only if all 4 gauge vectors $L^{(1)}$, $L^{(2)}$, $L^{(3)}$, $L^{(4)}$ from (10) obey the Lorentz condition. The fourth vector obeys this condition automatically because it was constricted as gradient-type solution $L_\alpha^{(4)} = \nabla_\alpha\Phi, \quad \nabla^\alpha\nabla_\alpha\Phi = 0$. It can be shown that the three remaining solutions satisfy this condition as well.

Verifying the guage solutions. The four calibration solutions Q, R, S, T can be decomposed in linear combinations of the above 6 solutions Ψ_i:

$$
\begin{aligned}
Q &= q_1\Psi_1 + q_2\Psi_2 + q_3\Psi_3 + q_4\Psi_4 + q_5\Psi_5 + q_6\Psi_6,\\
R &= r_1\Psi_1 + r_2\Psi_2 + r_3\Psi_3 + r_4\Psi_4 + r_5\Psi_5 + r_6\Psi_6,\\
S &= s_1\Psi_1 + s_2\Psi_2 + s_3\Psi_3 + s_4\Psi_4 + s_5\Psi_5 + s_6\Psi_6,\\
T &= t_1\Psi_1 + t_2\Psi_2 + t_3\Psi_3 + t_4\Psi_4 + t_5\Psi_5 + t_6\Psi_6.
\end{aligned} \tag{11}
$$

The coefficients of these linear combinations are given as follows:

$$
\begin{gathered}
q_1 = -i\sqrt{2}\lambda, \quad q_2 = 0, \quad q_3 = 0, \quad q_4 = i\frac{\lambda}{\sqrt{2}}, \quad q_5 = -i\frac{3\lambda^2 + 2k^2}{2\epsilon}, \quad q_6 = i\frac{k\lambda}{\sqrt{2}\epsilon},\\
r_1 = 0, \quad r_2 = 2ik, \quad r_3 = -\frac{i\lambda}{\sqrt{2}}, \quad r_4 = 0, \quad r_5 = -\frac{ik\lambda}{\sqrt{2}\epsilon}, \quad r_6 = -i\left(\frac{k^2}{\epsilon} + \epsilon\right),\\
s_1 = 0, \quad s_2 = 0, \quad s_3 = -ik, \quad s_4 = -i\frac{\lambda}{\sqrt{2}}, \quad s_5 = -i\frac{i\lambda^2}{2\epsilon}, \quad s_6 = -\frac{ik\lambda}{\sqrt{2}\epsilon},\\
t_1 = \frac{k^2}{\epsilon} - \epsilon, \quad t_2 = -\frac{2k^2}{\epsilon}, \quad t_3 = \frac{\sqrt{2}k\lambda}{\epsilon}, \quad t_4 = \epsilon - \frac{k^2}{\epsilon}, \quad t_5 = \sqrt{2}\lambda, \quad t_6 = 2k.
\end{gathered} \tag{12}
$$

Let us present relations (11) in matrix form:

$$
\begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{vmatrix} = C \begin{vmatrix} \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{vmatrix} + \Psi_1 \begin{vmatrix} -i\sqrt{2}\lambda \\ 0 \\ 0 \\ \dfrac{k^2}{\epsilon} - \epsilon \end{vmatrix} + \Psi_2 \begin{vmatrix} 0 \\ 2ik \\ 0 \\ -\dfrac{2k^2}{\epsilon} \end{vmatrix}, \tag{13}
$$

where the matrix C is given as

$$
C = \begin{vmatrix}
0 & i\dfrac{\lambda}{\sqrt{2}} & -i\dfrac{3\lambda^2 + 2k^2}{2\epsilon} & i\dfrac{k\lambda}{\sqrt{2}\epsilon}\\
-\dfrac{i\lambda}{\sqrt{2}} & 0 & -\dfrac{ik\lambda}{\sqrt{2}\epsilon} & -i\left(\dfrac{k^2}{\epsilon} + \epsilon\right)\\
-ik & -i\dfrac{\lambda}{\sqrt{2}} & -i\dfrac{i\lambda^2}{2\epsilon} & -\dfrac{ik\lambda}{\sqrt{2}\epsilon}\\
\dfrac{\sqrt{2}k\lambda}{\epsilon} & \epsilon - \dfrac{k^2}{\epsilon} & \sqrt{2}\lambda & 2k
\end{vmatrix}, \quad \lambda = \sqrt{\epsilon^2 - k^2}.
$$

Now let us turn to eq. (13) and multiply it by the inverse matrix C^{-1}, resulting in

$$
\begin{vmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_3 \\ \psi'_4 \end{vmatrix} = C^{-1} \begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{vmatrix} = \begin{vmatrix} \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{vmatrix} + \Psi_1 C^{-1} \begin{vmatrix} -i\sqrt{2}\lambda \\ 0 \\ 0 \\ \dfrac{k^2}{\epsilon} - \epsilon \end{vmatrix} + \Psi_2 C^{-1} \begin{vmatrix} 0 \\ 2ik \\ 0 \\ -\dfrac{2k^2}{\epsilon} \end{vmatrix}. \tag{14}
$$

Here, ψ'_i designate the new guage solutions related to the old ones by the linear transformation C. We readily find explicit form of the two last terms:

$$C^{-1}\begin{vmatrix} -i\sqrt{2}\lambda \\ 0 \\ 0 \\ \frac{k^2}{\epsilon}-\epsilon \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{2}\sqrt{\epsilon^2-k^2}}{k} \\ -\frac{2k^2}{k^2+\epsilon^2}-1 \\ 0 \\ \frac{2k\epsilon}{k^2+\epsilon^2}-\frac{\epsilon}{k} \end{vmatrix}, \quad C^{-1}\begin{vmatrix} 0 \\ 2ik \\ 0 \\ -\frac{2k^2}{\epsilon} \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{2k^2}{k^2+\epsilon^2} \\ 0 \\ -\frac{2k\epsilon}{k^2+\epsilon^2} \end{vmatrix}.$$

Therefore, the decomposition (14) can be presented as follows:

$$\begin{vmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_3 \\ \psi'_4 \end{vmatrix} = \begin{vmatrix} \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{vmatrix} + \Psi_1 \begin{vmatrix} \frac{\sqrt{2}\sqrt{\epsilon^2-k^2}}{k} \\ -\frac{2k^2}{k^2+\epsilon^2}-1 \\ 0 \\ \frac{2k\epsilon}{k^2+\epsilon^2}-\frac{\epsilon}{k} \end{vmatrix} + \Psi_2 \begin{vmatrix} 0 \\ \frac{2k^2}{k^2+\epsilon^2} \\ 0 \\ -\frac{2k\epsilon}{k^2+\epsilon^2} \end{vmatrix}. \tag{15}$$

From this presentation, we can conclude that Ψ_5 is a pure guage solution. Now, let us consider the decomposition of the three remaining guage solutions:

$$\begin{vmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_4 \end{vmatrix} = \begin{vmatrix} \Psi_3 \\ \Psi_4 \\ \Psi_6 \end{vmatrix} + \begin{vmatrix} \frac{\sqrt{2}\sqrt{\epsilon^2-k^2}}{k}\Psi_1 \\ \left(-\frac{2k^2}{k^2+\epsilon^2}-1\right)\Psi_1 \\ \left(\frac{2k\epsilon}{k^2+\epsilon^2}-\frac{\epsilon}{k}\right)\Psi_1 \end{vmatrix} + \begin{vmatrix} 0 \\ \frac{2k^2}{k^2+\epsilon^2}\Psi_2 \\ -\frac{2k\epsilon}{k^2+\epsilon^2}\Psi_2 \end{vmatrix}. \tag{16}$$

Hence, we have the following three separate relations:

$$\begin{gathered} \psi'_1 = \Psi_3 + \frac{\sqrt{2}\sqrt{\epsilon^2-k^2}}{k}\Psi_1 \Rightarrow \Psi_1 = \frac{k}{\sqrt{2}\sqrt{\epsilon^2-k^2}}(\psi'_1 - \Psi_3), \\ \psi'_2 = \Psi_4 + \left(-\frac{2k^2}{k^2+\epsilon^2}-1\right)\Psi_1 + \frac{2k^2}{k^2+\epsilon^2}\Psi_2, \\ \psi'_4 = \Psi_6 + \left(\frac{2k\epsilon}{k^2+\epsilon^2}-\frac{\epsilon}{k}\right)\Psi_1 - \frac{2k\epsilon}{k^2+\epsilon^2}\Psi_2. \end{gathered} \tag{17}$$

With the help of the first relation, one can eliminate the variable Ψ_1 from the remaining two equations

$$\psi'_2 = \Psi_4 + \left(-\frac{2k^2}{k^2+\epsilon^2}-1\right)\frac{k}{\sqrt{2}\sqrt{\epsilon^2-k^2}}(\psi'_1 - \Psi_3) + \frac{2k^2}{k^2+\epsilon^2}\Psi_2,$$

$$\psi'_4 = \Psi_6 + \left(\frac{2k\epsilon}{k^2+\epsilon^2}-\frac{\epsilon}{k}\right)\frac{k}{\sqrt{2}\sqrt{\epsilon^2-k^2}}(\psi'_1 - \Psi_3) - \frac{2k\epsilon}{k^2+\epsilon^2}\Psi_2.$$

The two last relations can be rewritten as follows:

$$\varphi_1' = \psi_2' - \alpha\psi_1' = \Psi_4 + (a\Psi_2 + b\Psi_3), \quad \varphi_2' = \psi_4' - \beta\psi_1' = \Psi_6 + (c\Psi_2 + d\Psi_3), \tag{18}$$

where the functions φ_1' and φ_2' are new guage solutions. The used numerical parameters are determined as

$$\alpha = -\frac{1}{\sqrt{2}}\left(1 + \frac{2k^2}{k^2+\epsilon^2}\right)\frac{k}{\sqrt{\epsilon^2-k^2}}, \quad a = \frac{2k^2}{k^2+\epsilon^2}, \quad b = \frac{1}{\sqrt{2}}\left(1 + \frac{2k^2}{k^2+\epsilon^2}\right)\frac{k}{\sqrt{\epsilon^2-k^2}};$$
$$\beta = -\frac{1}{\sqrt{2}}\frac{\epsilon^2-k^2}{k^2+\epsilon^2}\frac{\epsilon}{\sqrt{\epsilon^2-k^2}}, \quad c = -\frac{2k\epsilon}{k^2+\epsilon^2}, \quad d = \frac{1}{\sqrt{2}}\frac{\epsilon^2-k^2}{k^2+\epsilon^2}\frac{\epsilon}{\sqrt{\epsilon^2-k^2}}. \tag{19}$$

It follows from equation (34) that two combinations of the basic functions exist which do not contain any guage constituents. These are defined by the formulas

$$\Psi_1^{\text{phys}} = \Psi_4 - (a\Psi_2 + b\Psi_3), \quad \Psi_2^{\text{phys}} = \Psi_6 - (c\Psi_2 + d\Psi_3). \tag{20}$$

Solutions Ψ_1^{phys} and Ψ_2^{phys} refer to physical states that contribute to the energy-momentum tensor.

Conclusions. The main results are as follows. We have found 6 independent solutions of the system of equations, describing the massless spin 2 particle in 50-component approach. We have obtained explicit expressions for 4 guage solutions defined in accordance with the Pauli – Fierz general approach; they provide us with exact solutions of the system under consideration, and relate to states which do not contribute to physically observable quantities like energy-momentum tensor. Finally, we have constructed two classes of solutions which correspond to physically observable states. These results demonstrate correctness of the developed theory for the massless spin 2 field.

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