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Maxim M. Vaskouski, Hanna A. Zadarazhniuk, Aleksey U. Hanimar

Belarusian State University, Minsk, Republic of Belarus

ISOPERIMETRIC PROFILE AND RESISTANCE DIAMETERS OF CAYLEY GRAPHS ON SYMMETRIC GROUPS WITH COXETER GENERATING SETS

Abstract. Using combinatorial approaches we obtain asymptotically exact bounds on isoperimetric numbers and generalized edge-isoperimetric numbers of bubble-sort Cayley graphs. We show that the generalized edge-isoperimetric constants of the bubble-sort Cayley graph BS_n are equal to $\Theta(n)$ unless this constant is the Cheeger constant, otherwise we obtain asymptotic $\Theta\left(\frac{1}{n}\right)$. We apply these results to derive refined explicit estimates of resistance distance in bubble-sort Cayley graphs. It is proved that the resistance distance between any two vertices of BS_n is lower bounded by $\frac{2}{n}$ and upper bounded by $\frac{11+5\sqrt{5}+\varepsilon}{n}$ with arbitrarily small $\varepsilon > 0$ for all sufficiently large n.

Keywords: resistance distance, bubble-sort Cayley graph, isoperimetric inequalities, Cheeger constant

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М. М. Васьковский, А. О. Задорожнюк, А. В. Гонимар

Белорусский государственный университет, Минск, Республика Беларусь

ИЗОПЕРИМЕТРИЧЕСКИЙ ПРОФИЛЬ И РЕЗИСТИВНЫЕ ДИАМЕТРЫ ГРАФОВ КЭЛИ НА СИММЕТРИЧЕСКОЙ ГРУППЕ С ПОРОЖДАЮЩИМИ МНОЖЕСТВАМИ КОКСЕТЕРА

Аннотация. С использованием комбинаторных методов получены асимптотически точные оценки на изопериметрические числа и обобщенные реберно-изопериметрические числа графов Кэли пузырьковой сортировки. Показано, что обобщенные реберно-изопериметрические числа графа Кэли пузырьковой сортировки BS_n имеют асимптотику $\Theta(n)$, за исключением случая постоянной Чигера, которая, в свою очередь, имеет асимптотику $\Theta\left(\frac{1}{n}\right)$. Эти результаты применяются для вывода более точных оценок на резистивные расстояния в таких графах, а именно доказано, что резистивное расстояние между любыми двумя вершинами BS_n находится между $\frac{2}{n}$ и $\frac{11+5\sqrt{5}+\varepsilon}{n}$ для сколь угодно малого $\varepsilon > 0$ для всех достаточно больших n.

Ключевые слова: резистивное расстояние, граф Кэли пузырьковой сортировки, изопериметрические неравенства, постоянная Чигера

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Introduction. Discrete isoperimetric inequalities (DII) are widely used in many fields of mathematics and computer science. Such inequalities play crucial role in the analysis of expansion properties, graph connectivity in terms of resistance distance, percolation on graphs, some geometrical questions related to graph curvature [1–3]. Although there is a couple of generic DII, such as the Cheeger, Buser-type inequalities [4, 3], many important classes of graphs including Cayley graphs on symmetric groups with Coxeter generators are not covered properly by those. Sometimes we need to investigate deeper combinatorial structure of graphs to get exact isoperimetric bounds.

Bubble-sort Cayley graphs have a lot of applications for building inter-connection networks and analyzing some sorting algorithms [5–8].

One of the metrics which naturally arise when dealing with networks and signal transmission is hitting time between two vertices, that is the expected time it takes a random walk to travel from one of them to the other. The normalized version of this metric is known as a resistance distance. It is a convenient tool to encode the cluster structure of a network in many fields of computer science [9, 7], chemometrics and bioinformatics [10], mathematical physics [11]. This metric was introduced in paper [12] based on the Kirchhoff and Ohm laws in the corresponding electrical circuit. As opposed to the geodesic (shortest path) distance, the resistance distance between two vertices takes into account all paths between them.

The goal of this paper is to find exact asymptotic estimates for isoperimetric numbers of the bubble-sort Cayley graphs. Also we are interested in refine known bounds for the resistance distance in the bubble-sort Cayley graphs.

The main part of the paper has the following structure. In the third and the fourth sections we obtain asymptotically exact estimates on isoperimetric and generalized edge-isoperimetric numbers in the bubble-sort Cayley graphs. There are two core ideas which allow us to achieve this result: the first one is that any bubble-sort Cayley graph is a partial cube (can be isometrically embedded into some hypercube), the second one is to modify the bubble-sort Cayley graph a bit to make it edge-transitive. In the fifth section we use a modification of the flow method to connect concepts of generalized edge-isoperimetric numbers and resistance distance in generic graphs. Finally, with the help of the previous section's results we obtain refined estimates on the resistance distance in the bubble-sort Cayley graphs. In the sixth section we demonstrate how the established connection between generalized edge-isoperimetric numbers and resistance distance could give an alternative explanation of major difference between electrical circuits on two dimensional and higher dimensional grids.

Preliminaries. Let's remind key objects we will use further.

For a finite group Γ let T be a self-conjugate $(\forall t \in T \;\; \exists t^{-1} \in T)$ generating set without the identical element. The *Cayley graph* of Γ generated by T is an undirected graph Cay(Γ,T), where elements of Γ are vertices and two vertices g_1, g_2 are adjacent if and only if $g_2 g_1^{-1} \in T$.

For a finite graph $G = (V,E)$ let X be a non-empty subset of V. The set of edges $\partial X = \{(u,v) \in E : u \in X, v \in V \setminus X\}$ is called the *edge boundary* of the set X. Throughout the paper we will assume that $0 < |X| \le \frac{|V|}{2}$ considering edge boundaries.

Recall that graph $G = (V,E)$ is called *edge-transitive* if for any two edges e_1, $e_2 \in E$ there exists an automorphism of G that maps e_1 to e_2. For edge-transitive graphs the estimate below follows directly from Theorem 3.5 proved in [13].

P r o p o s i t i o n 1. *Let $G = (V,E)$ be a finite connected edge-transitive graph, $X \subset V$ and let r be the harmonic mean of its minimum and maximum degrees. Then the following isoperimetric inequality holds:*

$$\frac{|\partial X|}{|X|} \ge \frac{r}{2\mathrm{diam}(G)}.$$

Let's denote by BS_n the bubble-sort Cayley graph $\mathrm{Cay}(S_n, T_n)$, where T_n is the Coxeter generating set of a symmetric group S_n $(T_n = \{(i,i+1) \,|\, i = \overline{1,n-1}\})$. Also we define the modified bubble-sort Cayley graph $\widetilde{BS}_n = \mathrm{Cay}(S_n, \widetilde{T}_n)$, where $\widetilde{T}_n = T_n \cup \{(1,n)\}$.

Recall that a subgraph G of the graph H is said to be *isometric* if the distance between every pair of vertices in G is the same as the respective distance in H.

P r o p o s i t i o n 2 [14]. *The bubble-sort Cayley graph BS_n is an isometric subgraph of the hypercube $Q_{n(n-1)/2}$.*

P r o p o s i t i o n 3 [15]. *Let S be a non-empty vertex subset of a hypercube Q_n, (S,E_S) be the induced subgraph of Q_n. Then $|E_S| \le \sum_{i=1}^{|S|-1} h(i)$, where $h(i)$ denotes the number of 1's in the binary representation of i.*

D e f i n i t i o n 1. Let $G_n = (V_n, E_n)$ be a sequence of connected graphs with $|V_n| \to \infty$ as $n \to \infty$. We shall say that family (G_n) admits (α,β)-isoperimetric signature if the following conditions held:

– $\alpha = \sup\{\delta \in \mathbb{R} \mid \exists c_1 > 0 : |\partial X| \geq c_1 |X|^{\delta}\ \forall X \subset V_n\ \forall n \in \mathbb{N}\}$;

– $\beta = \sup\{\theta \in \mathbb{R} \mid \exists c_2 > 0 : |\partial X| \geq c_2 |X|^{\alpha} \log^{\theta} |X|\ \forall X \subset V_n\ \forall n \in \mathbb{N}\}$.

R e m a r k 1. To compare expansion properties of two infinite families of graphs (G_n) and $(G_{n'})$ with (α,β) and (α',β')-isoperimetric signature respectively, it is convenient to define the order for isoperimetric signature in the following way: $(\alpha,\beta) \succeq (\alpha',\beta')$ if there $\alpha \geq \alpha'$ holds and equality $\alpha = \alpha'$ implies that $\beta \geq \beta'$. In this case we will write $(G_n) \succeq (G_{n'})$. Particularly, if (G_n) forms a family of edge-expanders [1] and admits (α,β)-isoperimetric signature, then $(\alpha,\beta) \succeq (1,0)$.

E x a m p l e 1. Let G_n be a sequence of connected graphs with $|V_n| \to \infty$ as $n \to \infty$. It's clear that the best possible (α,β)-isoperimetric signature is for the family of complete graphs $K_n : (\alpha,\beta) = (2,0)$, the worst case is the family of paths $P_n : (\alpha,\beta) = (0,0)$.

For the family of hypercubes $Q_n = (V_n, E_n)$ we have $|\partial X| \geq |X|\ \forall X \subset V_n$ [16]. Also we can find subsets $X_n \subset V_n$ such that $|\partial X_n| = |X_n|$ and $|X_n| \to \infty$ as $n \to \infty$ (for example, we can choose X_n in such way that X_n induces the hypercube Q_{n-1}). That is why $(\alpha,\beta) = (1,0)$.

Let $B_n^d = (V_n, E_n)$, $n \geq 1$, be the sequence of graphs of d-dimensional rectangular grid of a size n in one dimension. According to the Bollobas and Leader results [17], for any $X \subset V_n$ there holds

$$|\partial X| \geq \min\left\{|X|^{1-1/d}\, r n^{d/r-1} : r = 1, 2, \ldots, n\right\}. \tag{1}$$

It follows from (1) that

$$|\partial X| \geq (2|X|)^{1-1/d}. \tag{2}$$

For any n we consider a subset X_n of V_n that induces $B_{[n/2]}^d$. It is clear that $|\partial X_n| = n^{d-1}$, $|X_n| = n^{d-1}[n/2]$. Thus, the bound (2) is sharp, and $(\alpha,\beta) = (1 - 1/d, 0)$.

D e f i n i t i o n 2. Let $G = (V,E)$ be a finite graph and $\delta \in (0,1]$. We call the generalized edge-isoperimetric number the following value

$$h_E(\delta) = \min_{X \subset V} \frac{|\partial X|}{|X|^{\delta}},$$

where minimum is taken over all subsets $X \subset V$ with $0 < |X| \leq \frac{|V|}{2}$. Constant $h_E(1)$ is known as isoperimetric number (or Cheeger constant) and will be denoted by h_E.

Bounds on the Cheeger constant in bubble-sort Cayley graphs.

R e m a r k 2. It is possible to obtain estimates of the isoperimetric number from the following Cheeger inequality

$$\frac{\sigma}{2} \leq h_E \leq \sqrt{2d\sigma},$$

where d is the degree of a graph (assuming its regularity), σ is the spectral gap (the smallest positive eigenvalue of the Laplacian matrix of a graph).

For some graphs (hypercubes, for example) this approach gives fine results. But for the bubble-sort Cayley graphs BS_n we obtain really poor estimates with this approach:

$$\frac{c_1}{n^2} \leq h_E \leq \frac{c_2}{\sqrt{n}} \tag{3}$$

for some positive constants c_1, c_2.

Unfortunately, the bubble-sort Cayley graphs have negative Ricci curvature unless $n \leq 3$ [18]. This does not allow to use a discrete Buser-type inequality [3] to derive better estimates for h_E comparing to (3).

Now we will obtain much better estimates on h_E in the bubble-sort Cayley graphs by using a combinatorial approach instead of spectral arguments.

Theorem 1. *For any $n \geq 3$ the isoperimetric number h_E of the bubble-sort Cayley graph BS_n satisfies the inequalities*

$$\frac{2}{3n} \leq h_E \leq \frac{2}{n}.$$

Proof. To prove the upper estimate on h_E it is enough to provide an example of the set $X \subset S_n$ with $|\partial X| / |X| = \frac{2}{n}$ (here and further we consider subsets of vertices X satisfying normal constraints coming from definition of isoperimetric number, i. e. $0 < X \leq |S_n| / 2$). Let X be the set of all permutations $\sigma \in S_n$ such that $\sigma^{-1}(1) < \sigma^{-1}(2)$. It's obvious that $|X| = n!/2$. Now let's see what edges between X and $S_n \setminus X$ are possible. To go out of X, an edge (σ,τ) should make 1 occur after $2 : \sigma^{-1}(1) < \sigma^{-1}(2)$ and $\tau^{-1}(1) > \tau^{-1}(2)$. And since in the generating set we only have transpositions swapping consecutive elements, these 1 and 2 should be neighbors swapped by a transposition. There are $(n-1)!$ such vertices in X, and for each of them only one edge swapping 1 and 2 is possible, so we have $|\partial X| = (n-1)!$. Then $h_E \leq \frac{(n-1)!}{n!/2} = \frac{2}{n}$.

Let's prove the lower bound on h_E. The bubble-sort graph is not edge-transitive. However, if we add the transposition $(1,n)$ to the generating set, the resulting Cayley graph which is the modified bubble-sort Cayley graph $\widetilde{BS}_n$ will be edge-transitive. It contains BS_n as a subgraph and their vertex sets are the same. We are going to prove the following lemma which states that the sizes of the edge-boundaries of the same vertex subset in BS_n and in $\widetilde{BS}_n$ differ no more than by a constant factor.

Lemma. *Let $X \subset S_n$ be a subset of vertices of BS_n (as well as $\widetilde{BS}_n$), and let ∂X and $\partial' X$ denote its edge-boundary in BS_n and $\widetilde{BS}_n$ respectively. Then $|\partial X| \geq |\partial' X| / 3$.*

Proof. Denote $s_i = (i, i+1)$, $i \in \{1, \ldots, n-1\}$, and $s_n = (1,n)$. Any edge $(\sigma, s_i\sigma)$, $\sigma \in S_n$ will be called an s_i-type edge.

To obtain ∂X from $\partial' X$, we must delete from it all s_n-type edges.

Both BS_n and $\widetilde{BS}_n$ are connected graphs, so for any pair of vertices u, v, connected by an s_n-type edge there must exist a path between them containing no such edges. There may even exist several such paths which do not intersect, but for our proof it will be enough to consider the path of edges of types s_1, s_2, ..., s_{n-2}, s_{n-1}, s_{n-2}, ... s_2, s_1, which we will denote by $p(u,v)$. Since $\partial' X$ is a cut-set in the graph $\widetilde{BS}_n$, any path in $\widetilde{BS}_n$ between $u \in X$ and $v \in S_n \setminus X$ contains an edge from $\partial' X$. So, if these vertices are connected by an s_n-type edge belonging to $\partial' X$, there must also exist an edge from $p(u,v)$ that belongs to $\partial' X$.

It may be possible that for several pairs of vertices $u_i, v_i = s_n u_i$ their respective paths $p(u_i,v_i)$ all share a common edge, and this is the edge that belongs to $\partial' X$. Let us show that three paths $p(u_i,v_i)$, $i = 1,2,3$, corresponding to different pairs (u_i,v_i) cannot have a common edge.

Let's suppose that several paths $p(u_i,v_i)$ have a common edge e of an s_j-type. Then if $1 < j < n-1$ the other edges from these paths which can be incident to this one can be of just two types: s_{j-1} and s_{j+1}. Note that all edges from one vertex are of different types, so there can be no more than two such edges incident to each endpoint of e, and thus, just no more than two different paths. If e is of an s_1-type, one of its endpoints has to be u_i or v_i. But, because there's only one edge of the s_n-type from each vertex, this would mean that the edges of the s_n-type connecting vertices u_i to vertices v_i are in fact the same edge, and all those pairs of vertices are the same pair. Finally, if e is of an s_{n-1}-type, only edges of an s_{n-2}-type can be incident to it, and so there can be no more than one path $p(u_i,v_i)$ containing e.

Thus, in $\partial' X$ one edge of the type different from s_n corresponds to at most two edges of the s_n-type, meaning $|\partial X| \geq |\partial' X| / 3$. The lemma is proved.

Now we can complete the proof of the theorem.

Since a diameter of the graph $\widetilde{BS}_n$ is $\left[\frac{n^2}{4}\right]$ [8], Proposition 1 implies that

$$\frac{|\partial' X|}{|X|} \geq \frac{2}{n} \tag{4}$$

for any subset X of S_n. Taking into account the lemma above and inequality (4), we obtain the required lower estimate on h_E. The theorem is proved.

Asymptotic estimates of generalized edge-isoperimetric constants. According to Theorem 1, for the family of bubble-sort Cayley graphs BS_n we have $h_E = \Theta\left(\frac{1}{n}\right)$. Our next step is to investigate the asymptotics of generalized edge-isoperimetric constants $h_E(\delta)$, $\delta \in (0,1)$, in the bubble-sort Cayley graphs.

Let G be a Cayley graph of a finite group. The growth function $V(G,\rho)$ of the Cayley graph G is the number of vertices in the ball $B(\rho)$ of radius ρ surrounding the identical element with respect to geodesic (shortest path) distance. Define the function $\varphi(G,k)$ as follows:

$$\varphi(G,k) = \inf\{\rho : V(G,\rho) \ge k\}.$$

Recall the following Coulhon – Saloff – Coste inequality in the Cayley graphs [2]:

$$\frac{|\partial_V X|}{|X|} \ge \frac{1}{2\varphi(G, 2|X|)}$$

for any vertex subset X of G, where $\partial_V X$ denotes the exterior vertex boundary of X.

Since $\operatorname{diam}(BS_n) = \frac{n(n-1)}{2}$ [19], the ball $B(\rho)$ contains BS_k for $k = \left[\sqrt{\rho}\right]$. By the Stirling formula we get

$$V(BS_n, \rho) \ge \left[\sqrt{\rho}\right]! \ge Ce^{\sqrt{\rho}}$$

for some $C > 0$ that does not depend on n and ρ.

Therefore,

$$\varphi(BS_n, k) \le \inf\left\{\rho : Ce^{\sqrt{\rho}} \ge k\right\} = \log^2 \frac{k}{C}.$$

Taking into account the Coulhon – Saloff – Coste inequality, we get the following estimate for the edge boundaries in BS_n:

$$|\partial X| \ge c\frac{|X|}{\log^2 |X|} \tag{5}$$

for all vertex subsets X of BS_n and some positive constant c that does not depend on n.

Since the graphs BS_n are bipartite and do not contain subgraphs $K_{2,3}$ [19], it follows [20] from Turan results on a maximal number of edges in bipartite graphs containing no complete bipartite subgraphs $K_{2,t}$ [21, 22] that

$$|\partial X| \ge c_1 n |X|^{1/2} \tag{6}$$

for all vertex subsets X of BS_n with $|X| = o(n^2)$ and some positive constant c_1, that does not depend on n.

Gathering together (5) and (6), we obtain that for any $\delta \in \left(0, \frac{1}{2}\right)$ for the graphs BS_n we have

$$h_E(\delta) = \Theta(n). \tag{7}$$

For the accurate analysis of resistance distance asymptotic it is important to extend the result (7) to some $\delta > \frac{1}{2}$. To make this extension possible we are going to derive better isoperimetric inequalities for subexponential volumes in BS_n comparing to (5) and (6).

T h e o r e m 2. *Let $n \ge 3$. For any vertex subset X of the bubble-sort Cayley graph BS_n and any $\alpha \in (0,1)$ there holds the inequality*

$$|\partial X| \ge f_n^{\alpha}(|X|), \text{ where } f_n^{\alpha}(x) = \begin{cases} \big((1-\alpha)n - 1\big)x, & \text{if } x \le 2^{\alpha n}, \\ \frac{2x}{3n}, & \text{if } x > 2^{\alpha n}. \end{cases}$$

P r o o f. If $|X|>2^{\alpha n}$, the needed estimate follows directly from Theorem 1. So, it remains to consider $|X|\le 2^{\alpha n}$. Let E_X be the edge set in the subgraph of BS_n induced by vertices in X. According to Proposition 2, BS_n is a subgraph of the hypercube $Q_{n(n-1)/2}$. Let's denote by $(X,\widetilde{E}_X)$ the subgraph of $Q_{n(n-1)/2}$ induced by vertices in X.

Let's apply Proposition 2. Then we have

$$|E_X|\le|\widetilde{E}_X|\le\sum_{i=1}^{|X|-1}h(i)\le\frac{|X|\log_2|X|}{2}. \tag{8}$$

With the help of estimate (8) we obtain the following bound for the cardinality of edge boundary in the bubble-sort Cayley graph:

$$|\partial X|=(n-1)|X|-2|E_X|\ge|X|(n-1-\log_2|X|)\ge|X|\big((1-\alpha)n-1\big).$$

The theorem is proved.

R e m a r k 3. It's clear that Theorem 2 provides asymptotically best possible isoperimetric estimate (up to constant multiplier) for subsets X when $\frac{1}{n}\log_2|X|$ is separated above from 1. Also Theorem 1 implies that the function f_n^{α} gives an exact estimate (up to constant multiplier) at $x=\frac{n!}{2}$.

R e m a r k 4. Note that sharp isoperimetric inequalities for the transposition Cayley graphs TS_n (generating set is the set of all transpositions (i,j) of S_n) were proved in paper [23].

Now we are ready to prove that $h_E(\delta)=\Theta(n)$ in BS_n for any $\delta\in(0,1)$.

T h e o r e m 3. *For any* $\delta\in(0,1)$ *and* $\varepsilon>0$ *there exists* $n_0(\delta,\varepsilon)$ *such that for any* $n\ge n_0(\delta,\varepsilon)$ *the generalized edge-isoperimetric numbers* $h_E(\delta)$ *of the bubble-sort Cayley graph* BS_n *satisfy the inequalities*

$$(1-\varepsilon)n\le h_E(\delta)\le n-1.$$

P r o o f. The upper bound is trivial (it is enough to consider any one-element subset of S_n).

Let's take arbitrary $\delta\in(0,1)$ and $\varepsilon\in(0,1)$. Let $X\subset S_n$, $0<|X|\le|S_n|/2$. It follows from Theorem 2 that

$$\frac{|\partial X|}{|X|^{\delta}}\ge\frac{2|X|^{1-\delta}}{3n}\ge n$$

assuming that $|X|^{1-\delta}\ge 1.5n^2$. Let's consider subsets X with $|X|^{1-\delta}\le 1.5n^2$.

One can find $n_0(\delta,\varepsilon)$ such that for any $n\ge n_0(\delta,\varepsilon)$ there holds

$$|X|\le 2^{\alpha n},$$

where $0<\alpha<\varepsilon$.

Then Theorem 2 implies

$$\frac{|\partial X|}{|X|^{\delta}}\ge\big((1-\alpha)n-1\big)|X|^{1-\delta}\ge(1-\varepsilon)n,$$

that is true for all sufficiently large values of n. The theorem is proved.

R e m a r k 5. Benjamini and Kozma [24] conjectured that $h_E(\delta)$ is separated from 0 below over all finite connected vertex transitive graphs G with a diameter less than $|G|^{1-\delta}$. Theorem 3 states that we have even $h_E(\delta)=\Theta(n)$ for the family of bubble-sort Cayley graphs.

Let's go back to isoperimetric inequality (5) and try to understand whether we can improve exponent 2 in logarithm. The following theorem shows that we can replace 2 by 1, and this exponent is the best possible.

T h e o r e m 4. *The family of the bubble-sort Cayley graphs* BS_n, $n\ge 3$, *admits* $(1,-1)$*-isoperimetric signature.*

P r o o f. It follows from Theorem 1 that there exists $c>0$ such that for all $n\ge 3$ and any $X\subset S_n$, $0<|X|\le|S_n|/2$ we have

$$|\partial X| \geq c\frac{|X|}{\log|X|}. \tag{9}$$

Indeed, if $|X| \leq 2^{\alpha n}$ for some $\alpha \in (0,1)$ that does not depend on n, inequality (9) clearly holds.

If $|X| > 2^{\alpha n}$, we get

$$|\partial X| \geq \frac{2}{3n}|X| \geq \frac{2\alpha}{3}\frac{|X|}{\log_2|X|}.$$

As it was shown in the proof of Theorem 1, there exists $X \subset S_n$ such that $|X| = n!/2$ and

$$\frac{|\partial X|}{|X|} = \frac{2}{n}. \tag{10}$$

Let the family BS_n admits (α,β)-isoperimetric signature. It follows from (9) and (10) that α = 1. It follows from (9) that β ≥ –1. Taking into account (10) and the Stirling formula we obtain that β ≤ –1. Hence, (α,β) = (1,–1). The theorem is proved.

R e m a r k 6. We conjecture that a stronger isoperimetric inequality comparing to (9) must be valid for the bubble-sort Cayley graphs:

$$|\partial X| \geq c\frac{|X|\log\log|X|}{\log|X|} \quad \forall X \subset S_n \quad \forall n \geq 3.$$

E x a m p l e 2. It is interesting to find an isoperimetric signature for other widely used families of Cayley graphs on symmetric groups, for example, to the family of star Cayley graphs SS_n (generating set is of transpositions (1,2), (1,3), …, (1,n)) and the family of transposition Cayley graphs TS_n.

Since the spectral gap is equal to 1 for any SS_n [25], the Cheeger inequality implies that $|\partial X| \geq |X|$ for any vertices subset X. Let n be even. One can find a set X such that $|\partial X| = |X|$ and $|X| = \frac{n!}{2}$ (we can push into X all permutations σ such that $\sigma(1) \leq \frac{n}{2}$). Hence, the family SS_n has (1,0)-isoperimetric signature.

It is known that spectral gap of TS_n is equal to n [26], so, the Cheeger inequality implies that $|\partial X| \geq n|X|$ for any $X \subset S_n$. Let's take X to be the set of all permutations $\sigma \in S_n$ such that $\sigma^{-1}(1) < \sigma^{-1}(2)$. For this subset X we have $|X| = \frac{n!}{2}$ and $|\partial X| = \frac{2n-1}{3}|X|$. Similarly to the proof of Theorem 4, application of the Stirling formula allows one to show that the family TS_n has (1,1)-isoperimetric signature.

Finally, in terms of Remark 1 we can make the following comparison:

$$(K_n) \succeq (TS_n) \succeq (SS_n) \succeq (Q_n) \succeq (BS_n) \succeq (B_n^d) \succeq (P_n).$$

R e m a r k 7. We conjecture that for any (α,β) such that $(2,0) \succeq (\alpha,\beta) \succeq (0,0)$ one can build a sequence of connected graphs $G_n = (V_n,E_n)$ with $|V_n| \to \infty$ that admits (α,β)-isoperimetric signature.

Application to the explicit estimate of resistance distance. In this section we apply the results obtained in the previous section to refine known bounds for the resistance distance in graphs. Recall that the resistance distance $R_{u,v}$ between two vertices u and v in a finite connected graph $G = (V,E)$ can be defined as $\frac{C_{u,v}}{2|E|}$, where $C_{u,v}$ is commute time between u and v (which is the expected length of a random walk from u to v and back) [27]. Also there is an equivalent definition of the resistance distance based on electrical circuit theory [12, 28].

T h e o r e m 5. *Let $G = (V,E)$ be a finite connected graph, d be the maximal vertex degree in it. Then for any $\delta \in \left(\frac{1}{2},1\right]$ such that $h_E(\delta) > 1$ and any u, $v \in V$ there hold the inequalities*

$$\frac{2}{d+1} \leq R_{u,v} \leq \frac{M_\delta d}{h_E^2(\delta)},$$

where

$$M_\delta = 2\inf\left\{\frac{\alpha^2(\gamma-1)\gamma^{2\delta-1}}{(\alpha-1)(\gamma^{2\delta-1}-1)} + \frac{\alpha\gamma^\delta}{\gamma^\delta-1} \,|\, \alpha \ge \frac{h_E(\delta)}{h_E(\delta)-1}, \gamma\in(1,2]\right\}.$$

P r o o f. Let's prove the lower bound. It follows from the monotonicity principle [29] that we can set all edge resistances to be 0, except edges from u and v. We replace all vertices except u and v by one vertex w. The obtained scheme is equivalent to a 3-node graph with multiple edges between u and w, w and v, and perhaps, an edge between u and v. If vertices u and v are non-adjacent in G_n, the resistance distance between u and v in the obtained 3-node scheme is at least $2/d$. Otherwise the resistance distance between u and v is at least $2/(d+1)$.

To prove the upper bound we will use the idea from the unpublished paper of Anari and Oveis Gharan (2014).

We fix arbitrary $\delta\in\left(\frac{1}{2},1\right]$, $\alpha\ge\frac{h_E(\delta)}{h_E(\delta)-1}$ and vertices u, $v\in V$. Let's assume that the current flow is 1 and the potential of the sink vertex v is $p_v = 0$. Denote the potential at the vertex $w\in V$ by p_w. So, to estimate the resistance $R_{u,v}$ we need to estimate the potential p_u. For any non-negative r let's define the sets with higher and lower potentials:

$$H_r = \{w\in V : p_w \ge r\}, \qquad L_r = \{w\in V : p_w \le r\}.$$

For any $X\subset V$, $|X|\le|V|/2$, we denote $f_\delta(X) = h_E(\delta)|X|^\delta$.

For some $r\ge 0$ such that $|H_r|\le|V|/2$ consider the flow through the edge boundary ∂H_r of H_r. Since ∂H_r is an edge-cutset separating vertices u and v, the flow can't be less than 1. The potentials of the vertices from H_r are higher than the potentials in $H_r^c = V\setminus H_r$, there cannot be any flow into H_r. Thus the flow between H_r and H_r^c is equal to 1:

$$\sum_{(s,t)\in\partial_E H_r,\, s\in H_r} y_{s,t} = 1,$$

where $y_{s,t}$ is the flow through the edge (s,t).

Then the average flow $\mathbb{E}(y_{s,t})$ over all edges of ∂H_r is equal to $1/|\partial H_r|$, where the expectation $\mathbb{E}(y_{s,t})$ is given with respect to the uniform distribution of edges (s,t) in ∂H_r. Markov's inequality implies that at least through $(1-\alpha^{-1})|\partial H_r|$ of edges in ∂H_r the flow $y_{s,t}$ is at most $\frac{\alpha}{f_\delta(H_r)}$:

$$\mathbb{P}\left(y_{s,t} < \frac{\alpha}{f_\delta(H_r)}\right) > 1 - \frac{f_\delta(H_r)}{\alpha}\mathbb{E}(y_{s,t}) = 1 - \frac{f_\delta(H_r)}{\alpha|\partial H_r|} \ge 1 - \frac{1}{\alpha}.$$

So one can find a subset $D_r\subseteq\partial H_r$ with at least $f_\delta(H_r)(1-\alpha^{-1})$ edges such that the current through them is no more than $\frac{\alpha}{f_\delta(H_r)}$.

Since the resistance of each edge is 1, the potentials difference on the endpoints of the edges of the set D_r is at most $\frac{\alpha}{f_\delta(H_r)}$ as well. Thus all these endpoints belong to $H_{r-\alpha/f_\delta(H_r)}$. Which means, there holds the inequality

$$|H_{r-\alpha/f_\delta(H_r)}| \ge |H_r| + \frac{f_\delta(H_r)(1-\alpha^{-1})}{d} \tag{11}$$

assuming that $|H_r|\le|V|/2$.

The same is to be done for lower potentials to obtain

$$|L_{r+\alpha/f_\delta(L_r)}| \ge |L_r| + \frac{f_\delta(L_r)(1-\alpha^{-1})}{d} \tag{12}$$

assuming that $|L_r|\le|V|/2$.

Take β > 0 such that $\gamma = 1 + \beta(1 - \alpha^{-1}) \in (1, 2]$. Applying recursively (11) and (12) $\left[\frac{\beta d\,|H_r|}{f_\delta(H_r)}\right] + 1$ and $\left[\frac{\beta d\,|L_r|}{f_\delta(L_r)}\right] + 1$ times respectively, we see that either there hold the inequalities

$$\left| H_{r - \frac{\alpha\beta d|H_r|}{f_\delta^2(H_r)} - \frac{\alpha}{f_\delta(H_r)}} \right| \ge \left| H_{r - \frac{\alpha}{f_\delta(H_r)}\left(\left[\frac{\beta d|H_r|}{f_\delta(H_r)}\right]+1\right)} \right| \ge |H_r| + \frac{f_\delta(H_r)(1-\alpha^{-1})}{d}\left(\left[\frac{\beta d\,|H_r|}{f_\delta(H_r)}\right] + 1\right) \ge \gamma\,|H_r|, \quad (13)$$

$$\left| L_{r + \frac{\alpha\beta d|L_r|}{f_\delta^2(L_r)} + \frac{\alpha}{f_\delta(L_r)}} \right| \ge \left| L_{r + \frac{\alpha}{f_\delta(L_r)}\left(\left[\frac{\beta d|L_r|}{f_\delta(L_r)}\right]+1\right)} \right| \ge |L_r| + \frac{f_\delta(L_r)(1-\alpha^{-1})}{d}\left(\left[\frac{\beta d\,|L_r|}{f_\delta(L_r)}\right] + 1\right) \ge \gamma\,|L_r|, \quad (14)$$

or the left-hand sides of (13) and (14) are greater than $|V|/\gamma \ge |V|/2$.

Taking $r = p_u$ and applying (13), we obtain that at least γ vertices have potentials higher or equal to

$$p_u - \frac{\alpha\beta d\,|H_r|}{f_\delta^2(H_r)} - \frac{\alpha}{f_\delta(H_r)} \ge p_u - \frac{\alpha\beta d}{h_E^2(\delta)} - \frac{\alpha}{h_E(\delta)} = r_1.$$

Analogously, at least γ^2 vertices have potentials higher or equal to

$$r_1 - \frac{\alpha\beta d\,|H_r|}{f_\delta^2(H_{r_1})} - \frac{\alpha}{f_\delta(H_{r_1})} \ge r_1 - \frac{\alpha\beta d}{h_E^2(\delta)\gamma^{2\delta-1}} - \frac{\alpha}{h_E(\delta)\gamma^{\delta}} = r_2.$$

Repeating this procedure at most $[\log_\gamma |V|]$ times, we obtain that at least $|V|/2$ vertices of G have potentials higher or equal to

$$p_u - \sum_{i=0}^{[\log_\gamma |V|]-1} \left(\frac{\alpha\beta d}{h_E^2(\delta)\gamma^{i(2\delta-1)}} + \frac{\alpha}{h_E(\delta)\gamma^{i\delta}} \right). \quad (15)$$

Analogously, starting with $|L_r| \ge 1$ for $r = p_v = 0$ and iterating inequality (14) at most $[\log_\gamma |V|]$ times, we obtain that at least $|V|/2$ vertices of G have potentials at most

$$p_v + \sum_{i=0}^{[\log_\gamma |V|]-1} \left(\frac{\alpha\beta d}{h_E^2(\delta)\gamma^{i(2\delta-1)}} + \frac{\alpha}{h_E(\delta)\gamma^{i\delta}} \right). \quad (16)$$

It follows from relations (15) and (16) that

$$p_u \le 2 \sum_{i=0}^{[\log_\gamma |V|]-1} \left(\frac{\alpha\beta d}{h_E^2(\delta)\gamma^{i(2\delta-1)}} + \frac{\alpha}{h_E(\delta)\gamma^{i\delta}} \right). \quad (17)$$

Since $h_E(\delta) \le d$, we have

$$p_u \le \frac{M_\delta d}{h_E^2(\delta)},$$

that finishes the proof of the theorem.

C o r o l l a r y 1. *For any ε > 0 there exists an effectively computable constant n_0 such that for any $n \ge n_0$ the resistance distance $R_{u,v}$ between any vertices u, v in the bubble-sort Cayley graph BS_n satisfies the following inequalities*

$$\frac{2}{n} \le R_{u,v} \le \frac{11 + 5\sqrt{5} + \varepsilon}{n}.$$

P r o o f. The lower bound follows immediately from Theorem 5. Let's consider the function

$$g(\alpha,\gamma,\delta) = \frac{2\alpha^2(\gamma-1)\gamma^{2\delta-1}}{(\alpha-1)(\gamma^{2\delta-1}-1)} + \frac{2\alpha\gamma^{\delta}}{\gamma^{\delta}-1}, \qquad \alpha > 1,\quad \gamma \in (1,2],\quad \delta \in (1/2, 1],$$

used for definition of the constant M_δ in Theorem 5 statement.

Since the function $g(\alpha,\gamma,\delta)$ is continuous and Theorem 3 holds, we obtain that for any $\varepsilon > 0$ one can find numbers $n_0 \in \mathbb{N}$ and $\delta \in \left(\frac{1}{2},1\right)$ such that $h_E(\delta) \geq (1-\varepsilon)n$ and $M_\delta \leq \widetilde{M}_1 + \varepsilon$ for all the bubble-sort Cayley graphs BS_n with $n \geq n_0$, where $\widetilde{M}_1 = \inf\{g(\alpha,\gamma,1) \mid \alpha > 1,\ \gamma \in (1,2]\}$.

It remains to show that $\widetilde{M}_1 = 11 + 5\sqrt{5}$. Indeed,

$$g(\alpha,\gamma,1) = \frac{2\alpha\gamma(\alpha\gamma-1)}{\alpha\gamma-(\alpha+\gamma)+1} \geq \frac{2\alpha\gamma(\alpha\gamma-1)}{(\sqrt{\alpha\gamma}-1)^2}.$$

By straightforward computations we obtain that the minimum of the right-hand side is achieved at $\sqrt{\alpha\gamma} = \frac{1+\sqrt{5}}{2}$. Hence, $\widetilde{M}_1 = g\left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, 1\right) = 11 + 5\sqrt{5}$. The corollary is proved.

R e m a r k 8. As it was shown in [29], the application of Kaneko – Suzuki algorithm [30] (for constructing $n-1$ node-to-node disjoint paths in the bubble-sort Cayley graph) gives asymptotically better upper bound on resistance distance in BS_n : $R_{u,v} = O(n)$ comparing to the spectral approach which gives only $O(n^2)$. In paper [20] there was suggested a flow-based approach to estimate the resistance distance in Cayley graphs, which gives right asymptotic $R_{u,v} = O(1/n)$ for BS_n, but the constant hidden in O is much worse comparing to Corollary 1.

C o r o l l a r y 2. *Let $G_n = (V_n, E_n)$ be a family of finite connected graphs with maximal degree $d_n \to \infty$ as $n \to \infty$. If there exists $\delta \in \left(\frac{1}{2},1\right]$ such that $h_E(\delta) = \Theta(d_n)$ in G_n, then $R_{u,v} = \Theta(1/d_n)$ for the resistance distances in G_n.*

E x a m p l e 3. Let B_n^d be the graph of d-dimensional rectangular grid of size n in one dimension. Taking into account Corollary 2 and Example 1, we obtain that $R_{u,v} = \Theta(1)$ for any fixed $d \geq 3$. If $d = 2$, Corollary 2 does not work, since $h_E(\delta) \to 0$ as $n \to \infty$ for any fixed $\delta \in \left(\frac{1}{2},1\right]$. Also we know from the results of [27] that $R_{u,v} = \Theta(\log n)$ in case $d = 2$.

Discussion. It is well-known [27] that good expansion properties (in terms of isoperimetric constant h_E) of a graph family lead to fine asymptotic estimates of the resistance distance. Informally speaking, in this case resistance distance $R_{u,v}$ is a local characteristic, it depends mainly on the degrees of vertices u, v (so called, Luxburg – Radl – Hein property) [9].

Also it was known that some families with poor expansion properties (finite grids in dimension $d \geq 3$, for example) admit Luxburg – Radl – Hein property. For the grids it was shown in [27] by using combinatorial arguments on flows that is very specific on graph structure.

In the present paper we have demonstrated that generalized isoperimetric constants $h_E(\delta)$ are much more flexible for the resistance distance analysis including graphs with complex combinatorial structure. To have Luxburg – Radl – Hein property in a family of graphs we rather need to look at asymptotic of $h_E(\delta)$ for some $\delta \in \left(\frac{1}{2},1\right)$ (Corollary 2). As we've already demonstrated, we have the needed asymptotic $h_E(\delta) = \Theta(d_n)$ for families of the bubble-sort Cayley graphs and d-dimensional grids with $d \geq 3$. This helps us to obtain the Luxburg – Radl – Hein property for both of these families using the same simple arguments (Theorem 5).

The threshold $\frac{1}{2}$ for δ in Theorem 5 is really important. Looking at the family of d-dimensional finite grids for $d = 2$ and $d = 3$, we see how the generalized isoperimetric constant $h_E(\delta)$ controls the asymptotic of the resistance distance in corresponding grids. Of course, this major difference between 2-dimensional and higher dimensional grids was well-known earlier (see [27]). The present paper provides a new kind of explanation to this phenomenon in terms of generalized isoperimetric constants.

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Information about the authors

Maksim M. Vaskouski – Dr. Sc. (Physics and Mathematics), Professor, Head of the Higher Mathematics Department, Belarusian State University (4, Nezavisimosti Ave., 220030, Minsk, Republic of Belarus). E-mail: vaskovskii@bsu.by

Hanna A. Zadarazhniuk – Ph. D. (Physics and Mathematics), Associate Professor of the Higher Mathematics Department Belarusian State University (4, Nezavisimosti Ave., 220030, Minsk, Republic of Belarus). E-mail: a_zadorozhnuyk@mail.ru

Aleksey U. Hanimar – Student, Belarusian State University (4, Nezavisimosti Ave., 220030, Minsk, Republic of Belarus).

Информация об авторах

Васьковский Максим Михайлович – доктор физико-математических наук, профессор, заведующий кафедрой ФМИС, Белорусский государственный университет (пр. Независимости, 4, 220030, Минск, Республика Беларусь). E-mail: vaskovskii@bsu.by

Задорожнюк Анна Олеговна – кандидат физико-математических наук, доцент кафедры ФМИС, Белорусский государственный университет (пр. Независимости, 4, 220030, Минск, Республика Беларусь). E-mail: a_zadorozhnuyk@mail.ru

Гонимар Алексей Владимирович – студент, Белорусский государственный университет (пр. Независимости, 4, 220030, Минск, Республика Беларусь).