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A MIXED PROBLEM FOR THE WAVE EQUATION IN A CURVILINEAR HALF-STRIP WITH DISCONTINUOUS INITIAL DATA

Abstract. We study a mixed problem for a one-dimensional wave equation in a curvilinear half-strip. The initial conditions have a discontinuity of the first kind at a single point. The mixed problem models the problem of a longitudinal impact on a finite elastic rod with a movable boundary. Using the method of characteristics, we obtain the solution in an explicit analytical form. For the problem in question, we prove the uniqueness of the solution and establish the conditions under which its classical solution exists.

Keywords: wave equation, mixed problem, method of characteristics, classical solution, matching conditions, conjugation conditions, discontinuous conditions, curvilinear domain

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СМЕШАННАЯ ЗАДАЧА ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ В КРИВОЛИНЕЙНОЙ ПОЛУПОЛОСЕ С РАЗРЫВНЫМИ НАЧАЛЬНЫМИ ДАННЫМИ

Аннотация. Изучается смешанная задача для одномерного волнового уравнения в криволинейной полуполосе. Начальные условия имеют разрыв первого рода в одной точке. Смешанная задача моделирует задачу о продольном ударе по конечному упругому стержню с подвижной границей. С использованием метода характеристик получено решение в явном аналитическом виде. Для рассматриваемой задачи доказывается единственность решения и устанавливаются условия, при которых существует ее классическое решение.

Ключевые слова: волновое уравнение, смешанная задача, метод характеристик, классическое решение, условия согласования, условия сопряжения, разрывные условия, криволинейная область

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1. Statement of the problem. In the curvilinear domain $Q=\{(t,x):t\in(0,\infty)\wedge x\in(\gamma(t),l)\}$, where l is a positive real number, of two independent variables $(t,x)\in\overline{Q}\subset\mathbb{R}^2$, for the wave equation

$$\left(\partial_t^2-a^2\partial_x^2\right)u(t,x)=f(t,x),\quad (t,x)\in Q, \tag{1}$$

we consider the following mixed problem with the initial conditions

$$u(0,x)=\varphi(x),\ \partial_t u(0,x)=\psi(x)+\begin{cases}0, & x\in[0,l),\\ v, & x=l,\end{cases}\ x\in[0,l], \tag{2}$$

and the boundary conditions

$$u\big(t,\gamma(t)\big)=\mu_1(t),\quad \left(\partial_t^2+b\partial_x\right)u(t,l)=\mu_2(t),\quad t\in[0,\infty), \tag{3}$$

where a, v, and b are real numbers; $a>0$ for definiteness; f is a function given on the set $\bar{Q}$; φ and ψ are some real-valued functions defined on the segment $[0,l]$; and μ_1 and μ_2 are some real-valued functions defined on the half-line $[0,\infty)$. We also assume that

$$\gamma\in C^1\big([0,\infty)\big),\quad \gamma'(t)\in(-a,a)\ \text{ for all }\ t\in[0,\infty),\quad \lim_{t\to+\infty}\gamma(t)\pm at=\pm\infty, \tag{4}$$

and the curves $x=\gamma(t)$ and $x=l$ do not intersect.

The mixed problem (1)–(4) models the following problem from the theory of longitudinal impact [1]. Suppose that an elastic finite homogeneous rod of constant cross-section, whose left moving boundary $x=\gamma(t)$ is fixed, is subjected at the initial moment $t=0$ to an impact at the end $x=l$ by a load that sticks to the rod. We also assume that an external volumetric force acts on the rod, that the displacements of the rod and the rate of their change at the initial moment $t=0$ are not equal to zero, and that there are no shock waves in the rod. Then, neglecting both the weight of the rod as a force and its possible vertical displacements, the displacements $u(t,x)$ of the rod satisfy the mixed problem (1)–(3), where $a=\sqrt{E\rho^{-1}}$, $b=SEM^{-1}$, where $E>0$ is Young's modulus of the rod material; $\rho>0$ is the density of the rod material; $S>0$ is the cross-sectional area of the rod; $M>0$ is the mass of the impacting load; $-v$ is the velocity of the impacting load; μ_2 is the external force acting on the end of the rod divided by the mass of the impacting load. The quantity $\mu_1(t)$ has a physical meaning of function that defines the movement of the end $x=0$ of the rod in the longitudinal direction. The function f is the external volumetric force divided by ρ.

In the case

$$\gamma(t)=0,\quad \mu_1=\mu_2\equiv 0,\quad \varphi=\psi\equiv 0,\quad f\equiv 0. \tag{5}$$

J. Boussinesq [2] constructed a formal solution of the problem (1)–(4) using the method of characteristics. This approach was developed in [3–8]. E. L. Nikolai [4] found a general expression for the solution of the problem (1)–(5) in the form of a piecewise given function using the Boussinesq method. S. I. Gaiduk [9] solved the problem (1)–(5) by the method of contour integration [10]. He strictly proved the existence and uniqueness of a unique generalized solution, but not its physical correctness. A similar mixed problem, but with a boundary condition $\left(\partial_t^2+b\partial_x+c\right)u(t,l)=0$ instead of $\left(\partial_t^2+b\partial_x\right)u(t,l)=0$, was studied in the work [11] by the method of contour integration, where again a unique generalized solution was constructed and its physical correctness was not justified. X. Yufeng and Z. Dechao [12] obtained a formal analytical solution in the form of a trigonometric series to a problem that is similar to a mixed problem (1)–(5), but with the boundary condition $u(t,0)-\beta\partial_x u(t,0)=0$ instead of $u(t,0)=0$. The problem (1)–(5) has also been solved using numerical methods, such as symbolic computations [13] and the finite element method [14].

When the data is smooth, $v=0$, and the half-strip is straight, i. e. $\gamma\equiv 0$, the problem (1)–(4) has been studied using Fourier series [3, 15, 16] and the method of characteristics [17]. Auxiliary issues related to the basis property of the system of functions appearing in the Fourier method for the problem (1)–(4) with $\gamma\equiv 0$ were studied in [18, 19]. Similar problems in curvilinear domains have been considered in the works [20–22]. Questions related to the stabilization and controllability of solutions to the wave equations in curvilinear domains have been studied in [23–25].

2. Curvilinear half-strip. Let us note some properties of the domain Q in which the problem is considered.

Assertion 1. *Let* $(t_0,x_0)\in Q$. *Then the value* x_0+at_0 *is nonnegative under the conditions* (4).

Assertion 2. *Let* $\alpha\in[0,\infty)$. *Then the equation* $\gamma(t)+at=\alpha$ *has a unique solution under the conditions* (4).

A s s e r t i o n 3. *Let* $\alpha \in (-\infty, 0]$. *Then the equation* $\gamma(t) - at = \alpha$ *has a unique solution under the conditions* (4).

A s s e r t i o n 4. *Let* $(t_0, x_0) \in Q$. *Then the curve* $\big(t, \gamma(t)\big)$ *intersects the line* $x + at = x_0 + at_0$ *at a single point under the conditions* (4).

A s s e r t i o n 5. *Let* $(t_0, x_0) \in Q$ *and* $x_0 - at_0 \le 0$. *Then the curve* $\big(t, \gamma(t)\big)$ *intersects the line* $x - at = x_0 - at_0$ *at a single point under the conditions* (4).

The p r o o f s of Assertions 1–5 are given in the article [21].

Consider the following functions:

$$\gamma_+ : [0, \infty) \ni t \mapsto \gamma(t) + at, \quad \gamma_- : [0, \infty) \ni t \mapsto \gamma(t) - at.$$

We also need the inverse of the functions γ_+ and γ_-, which will be denoted by the symbols Φ_+ and Φ_-, respectively, i. e. $\Phi_+\big(\gamma(t) + at\big) = t$ and $\Phi_-\big(\gamma(t) - at\big) = t$. Such functions exist by Assertions 2 and 3. From the inverse function theorem, we get the formulas:

$$\Phi'_-(t) = \frac{1}{\gamma'\big(\Phi_-(t)\big) - a}, \quad \Phi''_-(t) = -\frac{\gamma''\big(\Phi_-(t)\big)}{\big(\gamma'\big(\Phi_-(t)\big) - a\big)^3}, \quad t \in [0, \infty), \tag{6}$$

$$\Phi'_+(t) = \frac{1}{\gamma'\big(\Phi_+(t)\big) + a}, \quad \Phi''_+(t) = -\frac{\gamma''\big(\Phi_+(t)\big)}{\big(\gamma'\big(\Phi_+(t)\big) + a\big)^3}, \quad t \in [0, \infty). \tag{7}$$

Note that the representations (6) and (7) and condition (4) imply that Φ_+ is an increasing function and Φ_- is a decreasing function.

3. Auxiliary problem. Consider the following simple case

$$v = 0. \tag{8}$$

The solution u of the problem (1)–(4), (8) has the form

$$u(t, x) = w(t, x) + g(x - at) + p(x + at), \tag{9}$$

where w is a particular solution of Eq. (1). We can take it from the paper [20], it satisfies the homogeneous initial conditions

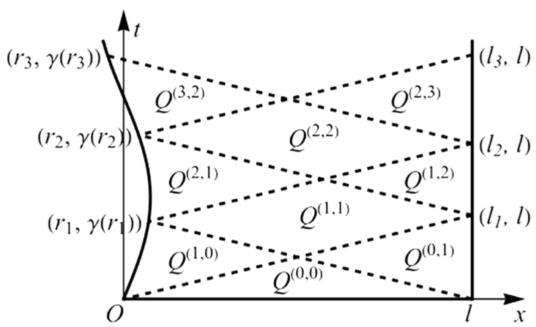
$$w(0, x) = \partial_t w(0, x) = 0, \quad x \in [0, l],$$

and belongs to the class $C^2(\overline{Q})$ if, for example, $f \in C^1(\overline{Q})$. Moreover, $\partial_t^2 w(0, x) = f(0, x)$ holds for all $x \in [0, l]$.

So, we want to find closed-form expressions for the functions g and p. To do this, we partition the domain $\overline{Q}$ according to the following formulas (for clarity see Figure):

$$Q^{(0,0)} = Q \cap \big\{(t, x) : x - at \in [0, l] \wedge x + at \in [0, l]\big\},$$

$$Q^{(1,0)} = Q \cap \big\{(t, x) : x - at \in [\gamma_-(r_1), 0] \wedge x + at \in [0, l]\big\},$$



Partitioning of the domain Q

$$Q^{(0,1)} = Q \cap \{(t,x) : x - at \in [0,l] \wedge x + at \in [l, l + al_1]\},$$

$$Q^{(i,j)} = Q \cap \{(t,x) : x - at \in [\gamma_-(r_i), \gamma_-(r_{i-1})] \wedge x + at \in [l + al_{j-1}, l + al_j]\}, \quad (10)$$

where $r_0 = l_0 = 0,\ l_i = r_{i-1} + a^{-1}(l - \gamma(r_{i-1})),\ r_i = \Phi_+(l + al_{i-1})$.

Let us demonstrate the correctness of the partitioning (10) of the domain Q. First, we draw the characteristic of the family $x - at = \text{const}$ through the point $(r_0 = 0, \gamma(r_0) = 0)$, which intersects the line $x = l$ at the point $(l_1 = la^{-1}, l)$. Next, we draw the characteristic of the family $x + at = \text{const}$ through the point $(l_1 = la^{-1}, l)$. According to Assertion 4, this characteristic intersects the curve $x = \gamma(t)$ at the point $(r_2, \gamma(r_2))$. At the same time, $r_2 > 0$. Next, we draw the characteristic of the family $x - at = \text{const}$ through the point $(r_2, \gamma(r_2))$. This characteristic intersects the line $x = l$ at the point (l_3, l) and, by Assertion 5, does not intersect the curve $x = \gamma(t)$ at any point other than $(r_2, \gamma(r_2))$. Thus, starting from $i = 0$, we sequentially define the points: (l_i, l), $(r_{i+1}, \gamma(r_{i+1}))$, (l_{i+2}, l), $(r_{i+3}, \gamma(r_{i+3}))$, (l_{i+4}, l), $(r_{i+5}, \gamma(r_{i+5}))$ etc.

Let us describe the process of finding points of the form

$$(l_{2i}, l), \quad (r_{2i+1}, \gamma(r_{2i+1})), \quad i = 0, 1, \ldots .$$

Through the point $(l_0 = 0, l)$ we draw the characteristic of the family $x + at = \text{const}$, which intersects the curve $x = \gamma(t)$ at one point $(r_1, \gamma(r_1))$ by virtue of Assertion 5. Now, through the point $(r_1, \gamma(r_1))$ we draw the characteristic of the family $x - at = \text{const}$, which intersects the line $x = l$ at one point (l_2, l) and, by Assertion 5, it does not intersect the curve $x = \gamma(t)$ at any point other than $(r_1, \gamma(r_1))$ according to Assertion 5. Then through the point (l_2, l) we draw the characteristic of the family $x + at = \text{const}$, which intersects the curve $x = \gamma(t)$ at one point $(r_3, \gamma(r_3))$ by virtue of Assertion 4. Thus, starting from $i = 0$, we sequentially define the points: $(r_i, \gamma(r_i))$, (l_{i+1}, l), $(r_{i+2}, \gamma(r_{i+3}))$, (l_{i+3}, l), $(r_{i+4}, \gamma(r_{i+4}))$ etc.

We will look for the function as a piecewise-defined, i. e.

$$u(t,x) = u^{(i,j)}(t,x), \quad (t,x) \in \overline{Q^{(i,j)}}. \quad (11)$$

Because of (9)–(11), we can write

$$u^{(i,j)}(t,x) = w(t,x) + g^{(i)}(x - at) + p^{(j)}(x + at), \quad (t,x) \in \overline{Q^{(i,j)}}. \quad (12)$$

We determine the functions $g^{(0)}$ and $p^{(0)}$ from the Cauchy conditions (2):

$$g^{(0)}(x) = \frac{\varphi(x)}{2} - \frac{1}{2a}\int_0^x \psi(\xi)d\xi + C_1, \quad x \in [0,l], \quad (13)$$

$$p^{(0)}(x) = \frac{\varphi(x)}{2} + \frac{1}{2a}\int_0^x \psi(\xi)d\xi - C_1, \quad x \in [0,l], \quad (14)$$

where C_1 is a real number. The function $g^{(i)}$ for $i \in \mathbb{N}$ can be defined from the Dirichlet boundary condition (3) on the curve $x = \gamma(t)$. We substitute (12), where $Q^{(i,j)} = Q^{(i,i-1)}$, into (3) and obtain

$$w(t, \gamma(t)) + g^{(i)}(\gamma(t) - at) + p^{(i-1)}(\gamma(t) + at) = \mu_1(t), \quad t \in [r_{i-1}, r_i], \quad i \in \mathbb{N}.$$

Changing the variable $z = \gamma(t) - at$, i. e. $t = \Phi_-(z)$, results in the equation

$$w(\Phi_-(z), \gamma(\Phi_-(z))) + g^{(i)}(z) + p^{(i-1)}(\gamma(\Phi_-(z)) + a\Phi_-(z)) = \mu_1(\Phi_-(z)), \quad \Phi_-(z) \in [r_{i-1}, r_i], \quad i \in \mathbb{N},$$

which we can solve to obtain

$$g^{(i)}(z) = \mu_1(\Phi_-(z)) - p^{(i-1)}(\gamma(\Phi_-(z)) + a\Phi_-(z)) - w(\Phi_-(z), \gamma(\Phi_-(z))),$$
$$\Phi_-(z) \in [r_{i-1}, r_i], \quad i \in \mathbb{N}. \quad (15)$$

The function $p^{(j)}$ for $j \in \mathbb{N}$ can be defined from the boundary condition (3) on the line $x = l$. Again, we substitute (12), where $Q^{(i,j)} = Q^{(j-1,j)}$, into (3) and get

$$a^2D^2g^{(j-1)}(l-at)+a^2D^2p^{(j)}(l+at)+$$
$$+b\left(Dg^{(j-1)}(l-at)+Dp^{(j)}(l+at)+\partial_x w(t,l)\right)+\partial_t^2 w(t,l)=\mu_2(t), \tag{16}$$
$$t\in[l_{i-1},l_i],\quad j\in\mathbb{N},$$

where D is the Newton – Leibniz operator. Changing the variable $t=a^{-1}(z-l)$ transforms Eq. (16) into

$$a^2D^2p^{(j)}(z)+bDp^{(j)}(z)=$$
$$=\mu_2\left(\frac{z-l}{a}\right)-a^2D^2g^{(j-1)}(2l-z)-bDg^{(j-1)}(2l-z)-b\partial_x v\left(\frac{z-l}{a},l\right)-\partial_t^2 v\left(\frac{z-l}{a},l\right), \tag{17}$$
$$z\in[l+al_{i-1},l+al_i],\quad j\in\mathbb{N}.$$

We solve Eq. (17) and obtain

$$p^{(j)}(z)=p^{(j)}(l+al_{j-1}+0)+\int\limits_{l+al_{j-1}}^{z}\exp\left(\frac{b(l+al_{j-1}-\eta)}{a^2}\right)\times$$
$$\times\left(Dp^{(j)}(l+al_{j-1}+0)+\int\limits_{l+al_{j-1}}^{\eta}a^{-2}\exp\left(\frac{b(\xi-l-al_{i-1})}{a^2}\right)\times\right. \tag{18}$$
$$\left.\times\left\{\mu_2\left(\frac{\xi-l}{a}\right)-a^2D^2g^{(j-1)}(2l-\xi)-bDg^{(j-1)}(2l-\xi)-b\partial_x v\left(\frac{\xi-l}{a},l\right)-\partial_t^2 v\left(\frac{\xi-l}{a},l\right)\right\}d\xi\right)d\eta,$$
$$z\in[l+al_{j-1},l+al_j],\quad j\in\mathbb{N}.$$

We choose the values $p^{(j)}(l+al_{j-1}+0)$ and $Dp^{(j)}(l+al_{j-1}+0)$ in the representation (18) by continuity, i. e.

$$p^{(j)}(l+al_{j-1}+0)=p^{(j-1)}(l+al_{j-1}-0),\quad Dp^{(j)}(l+al_{j-1}+0)=Dp^{(j-1)}(l+al_{j-1}-0),\quad j\in\mathbb{N}. \tag{19}$$

According to formulas (15), (18), and (19), the following relations hold for all $i\in\mathbb{N}$:

$$\Delta_g^i=g^{(i+1)}\left(\gamma_-(r_i)\right)-g^{(i)}\left(\gamma_-(r_i)\right)=p^{(i-1)}\left(\gamma_+(r_i)\right)-p^{(i)}\left(\gamma_+(r_i)\right)=0,$$

$$\tilde{\Delta}_g^i=Dg^{(i+1)}\left(\gamma_-(r_i)\right)-Dg^{(i)}\left(\gamma_-(r_i)\right)=-\frac{a+\gamma'(r_i)}{a-\gamma'(r_i)}\left(Dp^{(i-1)}\left(\gamma_+(r_i)\right)-Dp^{(i)}\left(\gamma_+(r_i)\right)\right)=0,$$

$$\approx{\Delta}_g^i=D^2g^{(i+1)}\left(\gamma_-(r_i)\right)-D^2g^{(i)}\left(\gamma_-(r_i)\right)=\frac{\left(a+\gamma'(r_i)\right)^2}{\left(a-\gamma'(r_i)\right)^2}\left(D^2p^{(i-1)}\left(\gamma_+(r_i)\right)-D^2p^{(i)}\left(\gamma_+(r_i)\right)\right),$$

$$\Delta_p^i=p^{(i+1)}(l+al_i)-p^{(i)}(l+al_i)=0,$$
$$\tilde{\Delta}_p^i=Dp^{(i+1)}(l+al_i)-Dp^{(i)}(l+al_i)=0,$$
$$\approx{\Delta}_p^i=D^2p^{(i+1)}(l+al_i)-D^2p^{(i)}(l+al_i)=-D^2g^{(i)}(l-al_i)+D^2g^{(i-1)}(l-al_i)-$$
$$-\frac{b}{a^2}Dg^{(i)}(l-al_i)+\frac{b}{a^2}Dg^{(i-1)}(l-al_i). \tag{20}$$

By virtue of the expressions $l_i=r_{i-1}+a^{-1}(l-\gamma(r_{i-1}))$ and $r_i=\Phi_+(l+al_{i-1})$ we have

$$\Delta_g^i=\tilde{\Delta}_g^i=\Delta_p^i=\tilde{\Delta}_p^{i-1}=0,\quad \approx{\Delta}_g^i=-\frac{\left(a+\gamma'(r_i)\right)^2}{\left(a-\gamma'(r_i)\right)^2}\approx{\Delta}_p^{i-1},\quad \approx{\Delta}_p^i=-\approx{\Delta}_g^{i-1}-\frac{b}{a^2}\tilde{\Delta}_g^{i-1},\quad i\in\mathbb{N}. \tag{21}$$

The base of the recurrence relations (21) can be computed using the representations (13), (14), (16), (18) and (19). So, after some simple calculations, we get

$$\Delta_g^0 = \delta_0 = \mu_1(0) - \varphi(0), \quad \tilde{\Delta}_g^0 = \delta_1 = \frac{\psi(0) + \gamma'(0)\varphi'(0) - \mu_1'(0)}{a - \gamma'(0)},$$

$$\tilde{\tilde{\Delta}}_g^i = \delta_2 = \frac{1}{(a-\gamma'(0))^3}\Big(\big(\mu_1'(0) - \psi(0)\big)\gamma''(0) - a^3\varphi''(0) + a^2\gamma'(0)\varphi''(0) + \gamma'(0)\times$$
$$\times\Big(f(0,0) + 2\gamma'(0)\psi'(0) + \gamma'(0)^2\varphi''(0) - \mu_1''(0)\Big) -$$
$$-a\Big(f(0,0) + 2\gamma'(0)\psi'(0) + \varphi'(0)\gamma''(0) + \gamma'(0)^2\varphi''(0) - \mu_1''(0)\Big),$$

$$\Delta_p^0 = \rho_0 = 0, \quad \tilde{\Delta}_p^0 = \rho_1 = 0, \quad \tilde{\tilde{\Delta}}_p^0 = \rho_2 = -\frac{f(0,l) - \mu_2(0) + b\varphi'v + a^2\varphi''(l)}{a^2}. \tag{22}$$

The following assertion holds.

A s s e r t i o n 6. *Let the smoothness conditions*

$$\varphi \in C^2([0,l]), \quad \psi \in C^1([0,l]), \quad \mu_1 \in C^2([0,\infty)), \quad \mu_2 \in C([0,\infty)), \quad \gamma \in C^2([0,\infty)), \quad f \in C^1(\overline{Q}) \tag{23}$$

be satisfied. Then the functions g and p, defined by the formulas (13)–(15), (18), (19), *and*

$$g(z) = g^{(0)}(z), \quad z \in [0,l], \quad g(z) = g^{(i)}(z), \quad \Phi_-(z) \in [r_{i-1}, r_i], \quad i \in \mathbb{N},$$
$$p(z) = p^{(0)}(z), \quad z \in [0,l], \quad p(z) = p^{(i)}(z), \quad z \in [l + al_{j-1}, l + al_j], \quad j \in \mathbb{N} \tag{24}$$

are twice continuously differentiable if and only if the following matching conditions are satisfied

$$\mu_1(0) - \varphi(0) = 0, \tag{25}$$

$$\mu_1'(0) - \psi(0) + \gamma'(0)\varphi'(0) = 0, \tag{26}$$

$$\mu_1''(0) - \Big(a^2 + \big(\gamma'(0)\big)^2\Big)\varphi''(0) - f(0,0) - 2\gamma'(0)\psi'(0) - \gamma''(0)\varphi'(0) = 0, \tag{27}$$

$$\mu_2(0) - f(0,l) - b\varphi'(0) - a^2\varphi''(l) = 0. \tag{28}$$

The p r o o f is based on the formulas (13)–(15), (18), (19), (21) and (22). Using the method of mathematical induction, we will first demonstrate that

$$g^{(i)} \in C^2\Big(\mathfrak{D}(g^{(i)})\Big), \quad p^{(i)} \in C^2\Big(\mathfrak{D}(p^{(i)})\Big), \quad i = 0,1,\dots .$$

Indeed, if $\varphi \in C^2([0,l])$, $\psi \in C^1([0,l])$, then, according to the expressions (13) and (14), $g^{(0)} \in C^2([0,l])$ and $p^{(0)} \in C^2([0,l])$. Thus, we have proved the base case of induction. Now suppose that

$$g^{(i)} \in C^2\Big(\mathfrak{D}(g^{(i)})\Big), \quad p^{(i)} \in C^2\Big(\mathfrak{D}(p^{(i)})\Big)$$

are true for some nonnegative integer i. In this case, if the conditions (23) are true, then according to the representations (15) and (18) we have

$$g^{(i+1)} \in C^2\big([\gamma_-(r_i), \gamma_-(r_{i+1})]\big), \quad p^{(i+1)} \in C^2([l + al_i, l + al_{i+1}]).$$

Therefore, we have proved the induction step. The function g is twice continuously differentiable everywhere except perhaps at the points $\gamma_-(r_i)$, $i = 0,1,\dots$, and the function p is twice continuously differentiable everywhere except perhaps at the points $l + al_i$, $i = 0,1,\dots$. For the functions g and p to be twice continuously differentiable everywhere in the domain of definition, it is necessary and sufficient that the following conditions are satisfied:

$$\Delta_g^i = \tilde{\Delta}_g^i = \tilde{\tilde{\Delta}}_g^i = \Delta_p^i = \tilde{\Delta}_p^i = \tilde{\tilde{\Delta}}_p^i = 0, \quad i = 0,1,\dots . \tag{29}$$

According to the discontinuity representations (21) and (22), the conditions (29) are true if and only if the matching conditions (25)–(28) are satisfied. The assertion is proved.

Assertion 7. *The functions g and p defined by the formulas* (13)–(15), (18), (19) *and* (24) *are of the form* $g = \tilde{g} + C_1$, $p = \tilde{p} - C_1$, *where* $\tilde{g}$ *and* $\tilde{p}$ *are some functions not depending on the constant* C_1.

Proof. We will prove the theorem using the method described in [26, p. 179]. We will apply the method of mathematical induction. According to the formulas (13)–(15), the expression holds

$$g^{(0)}(x) = \tilde{g}^{(0)}(x) + C_1, \quad p^{(0)}(x) = \tilde{p}^{(0)}(x) - C_1, \quad x \in [0,l],$$

where

$$\tilde{g}^{(0)}(x) = \frac{\varphi(x)}{2} - \frac{1}{2a}\int_0^x \psi(\xi)d\xi, \quad x \in [0,l],$$

$$\tilde{p}^{(0)}(x) = \frac{\varphi(x)}{2} + \frac{1}{2a}\int_0^x \psi(\xi)d\xi, \quad x \in [0,l].$$

Thus, the base case of induction is proved. Now, let us assume that the relation

$$g^{(i)} = \tilde{g}^{(i)} + C_1, \quad p^{(i)} = \tilde{p}^{(i)} - C_1, \tag{30}$$

is true for some $i \in \{0\} \cup \mathbb{N}$. According to the formula (15), we have the following representation:

$$g^{(i+1)}(z) = \tilde{g}^{(i+1)}(z) + C_1, \quad \Phi_-(z) \in [r_{i-1}, r_i],$$

where

$$\tilde{g}^{(i+1)}(z) = \mu_1\left(\Phi_-(z)\right) - \tilde{p}^{(i)}\left(\gamma\left(\Phi_-(z)\right) + a\Phi_-(z)\right) - w\left(\Phi_-(z), \gamma\left(\Phi_-(z)\right)\right) + C_1, \quad \Phi_-(z) \in [r_i, r_{i+1}].$$

In turn, the formulas (18) and (19) imply that

$$p^{(i+1)}(z) = \tilde{p}^{(i+1)}(z) + C_1, \quad \Phi_-(z) \in [l + al_i, l + al_{i+1}],$$

where

$$\tilde{p}^{(i+1)}(z) = \tilde{p}^{(i)}(l + al_i + 0) + \int_{l+al_i}^{z} \exp\left(\frac{b(l + al_i - \eta)}{a^2}\right) \times$$

$$\times\left(D\tilde{p}^{(i+1)}(l + al_i + 0) + \int_{l+al_i}^{\eta} \exp\left(\frac{b(\xi - l - al_i)}{a^2}\right) a^{-2}\left\{\mu_2\left(\frac{\xi - l}{a}\right) - a^2 D^2 \tilde{g}^{(i)}(2l - \xi) - bD\tilde{g}^{(i)}(2l - \xi) -\right.\right.$$

$$\left.\left.- b\frac{\partial w}{\partial x}\left(\frac{\xi - l}{a}, l\right) - \frac{\partial^2 w}{\partial t^2}\left(\frac{\xi - l}{a}, l\right)\right\} d\xi\right) d\eta, \quad z \in [l + al_i, l + al_{i+1}].$$

Thus, the induction step is proven. Consequently, the expression (30) holds for all $i \in \{0\} \cup \mathbb{N}$. Therefore, we can define the functions g and p as follows

$$g = \tilde{g} + C_1, \quad p = \tilde{p} - C_1,$$

where

$$\tilde{g}(z) = \tilde{g}^{(0)}(z), \quad z \in [0,l],$$

$$\tilde{g}(z) = \tilde{g}^{(i)}(z), \quad \Phi_-(z) \in [r_{i-1}, r_i], \quad i = 1, 2, \ldots,$$

$$\tilde{p}(z) = \tilde{p}^{(0)}(z), \quad z \in [0,l],$$

$$\tilde{p}(z) = \tilde{p}^{(j)}(z), \quad z \in [l + al_{j-1}, l + al_j], \quad j = 1, 2, \ldots .$$

The assertion has been proven.

Theorem 1. *Let the smoothness conditions* (23) *be satisfied. The mixed problem* (1)–(4), (8) *has a unique solution in the class* $C^2(\overline{Q})$ *if and only if the matching conditions* (25)–(28) *are satisfied. This solution is determined by the formulas* (11)–(15), (18), (19).

P r o o f. Assertions 6 and 7 imply the existence of a solution. The uniqueness of the solution stems from its construction and Assertion 7 because it is derived from the general solution. Assuming there are two solutions to the problem (1)–(4), (8) implies that the difference between them satisfies the homogeneous equation (1) and the homogeneous conditions (3) and (4). Formulas (11)–(15), (18), and (19), as well as Assertion 7, show that the homogeneous problem has only a zero solution. These results prove the uniqueness of the solution to the problem (1)–(4).

4. Main problem. Since in the general case $\psi \notin C^1([0,l])$, the problem (1)–(4) has no solution belonging to the class $C^2(\bar{Q})$, i. e. the problem (1)–(4) has no global classical solution defined on the set $\bar{Q}$. However, it is possible to define a classical solution on a smaller set $\bar{Q} \setminus \Gamma$ that will satisfy Eq. (1) on the set $\bar{Q} \setminus \Gamma$ in the standard sense and some additional conjugation conditions on the set Γ.

D e f i n i t i o n. A function u is a classical solution of the problem (1)–(4) if it is representable in the form $u = u_1 + u_2$, where u_1 is a classical solution of the problem (1)–(4) with $v = 0$ and u_2 satisfying the Eq. (1) with $f \equiv 0$, the initial conditions $u_2(0,x) = \partial_t u_2(0,x) = 0$, $x \in [0,l]$, the boundary conditions (3) with $\mu_1 = \mu_2 \equiv 0$, and the following matching conditions

$$[(u_2)^+ - (u_2)^-](t, x = \gamma_-(r_i) + at) = 0, \quad i = 0,1,..., \tag{31}$$

$$[(u_2)^+ - (u_2)^-](t, x = l + al_i - at) = 0, \quad i = 0,1,..., \tag{32}$$

$$\begin{gathered} [(\partial_t u_2)^+ - (\partial_t u_2)^-](t, x = l + al_i - at) = C^{(i)}, \quad i \in \mathrm{Even}[\mathbb{N} \cup \{0\}], \\ [(\partial_t u_2)^+ - (\partial_t u_2)^-](t, x = l + al_i - at) = 0, \quad i \in \mathrm{Odd}[\mathbb{N}], \end{gathered} \tag{33}$$

where $C^{(i)}$, $i = 0,1,...$, are some constants;

$$\mathrm{Even}[\Omega] = \{x : x \in \Omega \wedge x \equiv 0 \ (\mathrm{mod}\ 2)\},$$

and

$$\mathrm{Odd}[\Omega] = \{x : x \in \Omega \wedge x \equiv 1 \ (\mathrm{mod}\ 2)\}.$$

T h e o r e m 2. *Let the smoothness conditions* (23) *be satisfied. The mixed problem* (1)–(4) *has a unique solution in the sense of Definition if and only if the matching conditions* (25)–(28) *are satisfied.*

P r o o f. According to Theorem 1, under the smoothness conditions (23), the "smooth" part of the solution, i. e. the function u_1 from Definition, exists and is unique if and only if the conditions (25)–(28) are satisfied. The "discontinuous" part of the solution, i. e. the function u_2 from Definition, can be defined by the formula

$$u_2(t,x) = g_*^{(i)}(x - at) + p_*^{(j)}(x + at), \quad (t,x) \in \overline{Q^{(i,j)}}, \tag{34}$$

where

$$g_*^{(0)}(x) = p_*^{(0)}(x) = 0, \quad x \in [0,l], \tag{35}$$

$$g_*^{(i)}(z) = -p_*^{(i-1)}\big(\gamma(\Phi_-(z)) + a\Phi_-(z)\big), \quad \Phi_-(z) \in [r_{i-1}, r_i], \quad i \in \mathbb{N}, \tag{36}$$

$$\begin{gathered} p_*^{(j)}(z) = p_*^{(j-1)}(l + al_{j-1} - 0) + \int_{l+al_{j-1}}^{z} \exp\left(\frac{b(l + al_{j-1} - \eta)}{a^2}\right)\left\{Dp_*^{(j-1)}(l + al_{j-1} - 0) + \frac{C^{(j-1)}\chi_{\mathrm{Odd}}(j)}{a} - \right. \\ \left. - \int_{l+al_{j-1}}^{\eta} a^{-2} \exp\left(\frac{b(\xi - l - al_{i-1})}{a^2}\right)\left(a^2 D^2 g_*^{(j-1)}(2l - \xi) + bDg_*^{(j-1)}(2l - \xi)\right)d\xi\right\}d\eta, \\ z \in [l + al_{j-1}, l + al_j], \quad j \in \mathbb{N}, \end{gathered} \tag{37}$$

where χ_{Odd} is an indicator function of a set $\mathrm{Odd}[\mathbb{N}]$. The formulas (34)–(37) can be derived in the same way as (11)–(15), (18) and (19). Let us write this out in more detail. First, we construct the function u_2 from the general solution of Eq. (1) with $f \equiv 0$. It has the form

$$u_2(t,x) = \tilde{g}_*^{(i)}(x-at) + \tilde{p}_*^{(j)}(x+at), \quad (t,x) \in \overline{Q^{(i,j)}}, \quad i = 0,1,\ldots, \quad j = 0,1,\ldots, |i-j| \le 1. \tag{38}$$

Here, the functions $\tilde{g}_*^{(i)}$ and $\tilde{p}_*^{(i)}$ $(i = 0,1,\ldots)$ are piecewise twice continuously differentiable functions. From the homogeneous Cauchy conditions (2) with $\varphi \equiv \psi \equiv 0$ it follows [26, p. 174–175]

$$\tilde{g}_*^{(0)}(x) = \tilde{C}_1, \quad x \in [0,l], \tag{39}$$

$$\tilde{p}_*^{(0)}(x) = -\tilde{C}_1, \quad x \in [0,l], \tag{40}$$

where $\tilde{C}_1$ is a real integration constant. The functions $\tilde{g}_*^{(i)}$ for all $i \in \mathbb{N}$ are determined from the Dirichlet boundary condition (3) with $\mu_1 \equiv 0$. For a fixed positive integer i and $j = i-1$, we substitute the general solution (38) of Eq. (1) with $f \equiv 0$ into the boundary condition (3) with $\mu_1 \equiv 0$ and obtain

$$\tilde{g}_*^{(i)}\big(\gamma(t) - at\big) + \tilde{p}_*^{(i-1)}\big(\gamma(t) + at\big) = 0, \quad t \in [r_{i-1}, r_i], \quad i = 0,1,\ldots .$$

Using the results of Section 3, we arrive at the following equality:

$$\tilde{g}_*^{(i)}(z) = -\tilde{p}_*^{(i-1)}\big(\gamma(\Phi_-(z)) + a\Phi_-(z)\big), \quad \Phi_-(z) \in [r_{i-1}, r_i], \quad i = 1,2,\ldots . \tag{41}$$

Similarly, we define the function $\tilde{p}^{(j)}$ for all $j \in \mathbb{N}$:

$$\tilde{p}_*^{(j)}(z) = \tilde{p}_*^{(j)}(l + al_{j-1} + 0) + \int_{l+al_{j-1}}^{z} \exp\left(\frac{b(l + al_{j-1} - \eta)}{a^2}\right) \times$$

$$\times \left(D\tilde{p}_*^{(j)}(l + al_{j-1} + 0) + \int_{l+al_{j-1}}^{\eta} \exp\left(\frac{b(\xi - l - al_{i-1})}{a^2}\right) a^{-2} \times \right.$$

$$\left. \times \left\{ \mu_2\left(\frac{\xi - l}{a}\right) - a^2 D^2 \tilde{g}_*^{(j-1)}(2l - \xi) - bD\tilde{g}_*^{(j-1)}(2l - \xi) \right\} d\xi \right) d\eta, \quad z \in [l + al_{j-1}, l + al_j], \quad j = 1,2,\ldots . \tag{42}$$

Here the values $\tilde{p}_*^{(j)}(l + al_{j-1} + 0)$ and $D\tilde{p}_*^{(j)}(l + al_{j-1} + 0)$ in the representation (42) must be chosen so that the matching conditions (32) and (33) are satisfied. From the representation (38) it follows

$$[(u_2)^+ - (u_2)^-](t, x = l + al_j - at) = \tilde{p}_*^{(j+1)}(l + al_j + 0) - \tilde{p}_*^{(j-1)}(l + al_j - 0), \quad j = 0,1,\ldots,$$

$$\left[\left(\frac{\partial u_2}{\partial t}\right)^+ - \left(\frac{\partial u_2}{\partial t}\right)^- \right](t, x = l + al_j - at) = aD\tilde{p}_*^{(j+1)}(l + al_j + 0) - aD\tilde{p}_*^{(j)}(l + al_j + 0), \quad i = 0,1,\ldots .$$

So, we need to choose

$$\tilde{p}_*^{(j)}(l + al_{j-1} + 0) = \tilde{p}_*^{(j-1)}(l + al_{j-1} - 0), \quad j = 1,2,\ldots,$$

$$D\tilde{p}_*^{(j)}(l + al_{j-1} + 0) = D\tilde{p}_*^{(j-1)}(l + al_{j-1} - 0) + a^{-1}C^{(j-1))}\chi_{\mathrm{Odd}}(j), \quad j = 1,2,\ldots .$$

Similarly to Assertion 7, it can be shown that the functions $\tilde{g}_*^{(j)}$ and $p_*^{(j)}$ have the form

$$\tilde{g}_*^{(j)} = g_*^{(j)} + \tilde{C}_1, \quad \tilde{p}_*^{(j)} = p_*^{(j)} - \tilde{C}_1, \quad j = 0,1,\ldots,$$

where $g_*^{(j)}$ and $p_*^{(j)}$ are some functions that do not depend on the constant $\tilde{C}_1$ and are determined by the relations (35)–(37). Therefore, since the expression (38) does not depend on the constant $\tilde{C}_1$ then the constructed solution is unique and it is expressed by the formulas (38)–(42) when $\tilde{C}_1 = 0$, i. e. by the formulas (35)–(37).

The fulfillment of the conjugation conditions (31)–(33) is verified directly. Let us demonstrate this with the condition (31) for $i = 1$, i. e.

$$[(u_2)^+ - (u_2)^-](t, x = at) = 0. \tag{43}$$

The formulas (34)–(36) imply the representations $(u_2)^+(t, x = at) = 0$ and $(u_2)^-(t, x = at) = 0$. These equalities yield $[(u_2)^+ - (u_2)^-](t, x = at) = 0$. It proves the condition (43). The remaining conjugation conditions are verified similarly. The fact that the functions $g_*^{(i)}$ and $p_*^{(i)}$, $i = 0,1,...,$ belong to classes $C^2\left(\mathfrak{D}(g_*^{(i)})\right)$ and $C^2\left(\mathfrak{D}(p_*^{(i)})\right)$, respectively, as it is established analogously to Assertion 6, allows us to conclude that the function u_2 belongs to the class $C^2\left(\overline{Q^{(i,j)}}\right)$ for all $i \in \{0\} \cup \mathbb{N}$, $j \in \{0\} \cup \mathbb{N}$, $|i - j| \le 1$, and satisfies Eq. (1) on the sets $\overline{Q^{(i,j)}}$ with $f \equiv 0$. The fulfillment of the initial and boundary conditions also follows from the construction, since the functions $g_*^{(i)}$ and $p_*^{(i)}$, $i = 0,1,...,$ are chosen so that the homogeneous initial and boundary conditions are satisfied. The theorem is proved.

R e m a r k 1. The solution to problems (1)–(4) is not uniquely defined. Specifically, we must specify the constants $C^{(i)}$ for all $i \in \text{Even}[\mathbb{N}]$.

R e m a r k 2. According to Theorem 2, any choice of the constants $C^{(i)}$, $i \in \text{Even}[\mathbb{N}]$, uniquely determines the solution.

5. Physically correct solution. By a physically correct solution we mean one that correctly describes the impact process. The following statement holds.

A s s e r t i o n 8. *Suppose* $D\gamma(r_j) = 0$ *for all* $i \in \text{Even}[\mathbb{N}]$. *Then, we can set* $C^{(i)} = v$ *and obtain a physically correct solution in Theorem 2.*

The p r o o f of Assertion 8 is given in the paper [27].

The following theorem describes a more general case of Assertion 8.

T h e o r e m 3. *If we set*

$$C^{(i)} = v\prod_{j=1}^{i/2} \frac{a + \gamma'(r_{2j-1})}{a - \gamma'(r_{2j-1})}, \quad i \in \text{Even}[\mathbb{N} \cup \{0\}], \tag{44}$$

then a solution of the problem (1)–(4) *constructed in Theorem 2 is physically correct.*

P r o o f. The conditions (30) and (31) are derived from the continuity. Therefore, we only need to show the correctness of the condition (32) under the conditions. At the initial moment $t = 0$, the rod is subjected to an impact at the end $x = l$. It generates the shock wave that spreads along the characteristic $x + at = l$. Its velocity must satisfy the following condition

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = l - at) = v.$$

It proves (44) for $i = 0$. For the derivation of the previous equality, we refer the reader to our paper [28].

Furthermore, when the rod reaches its endpoint, it is immediately reflected and propagates along the characteristic at a speed that we cannot set but can calculate as follows:

$$[(\partial_t u)^+ - (\partial_t u)^-]\big(t, x = \gamma_-(r_1) - at\big) = v\frac{a + \gamma'(r_1)}{a - \gamma'(r_1)}.$$

Interacting with the moving end of the rod changes its velocity. Since waves propagate at the same speed in elastic rods [29], we should have

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = l + al_2 - at) = [(\partial_t u)^+ - (\partial_t u)^-]\big(t, x = \gamma_-(r_1) - at\big) = v\frac{a + \gamma'(r_1)}{a - \gamma'(r_1)}.$$

It proves (44) for $i = 2$.

We will prove it further using the method of mathematical induction. The base case has been proven. Now, let us prove the induction step. Assume that we have

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = l + al_i - at) = v\prod_{j=1}^{i/2} \frac{a + \gamma'(r_{2j-1})}{a - \gamma'(r_{2j-1})}$$

for some $i \in \text{Even}[\mathbb{N}]$. The wave that moves along the characteristic $l + al_i - at$ will be reflected from the end $x = \gamma(t)$ of the rod and its speed will become equal to

$$[(\partial_t u)^+ - (\partial_t u)^-]\big(t, x = \gamma_-(r_{i+1}) - at\big) = \nu \frac{a + \gamma'(r_{i+1})}{a - \gamma'(r_{i+1})} \prod_{j=1}^{i/2} \frac{a + \gamma'(r_{2j-1})}{a - \gamma'(r_{2j-1})} = \nu \prod_{j=1}^{1+i/2} \frac{a + \gamma'(r_{2j-1})}{a - \gamma'(r_{2j-1})}$$

according to the formulas (34)–(37). The wave propagates at the same speed in elastic rods, so

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = l + al_{i+2} - at) = [(\partial_t u)^+ - (\partial_t u)^-]\big(t, x = \gamma_-(r_{i+1}) - at\big) = \nu \prod_{j=1}^{1+i/2} \frac{a + \gamma'(r_{2j-1})}{a - \gamma'(r_{2j-1})}.$$

The induction step is proven. Thus, we have proven the physical correctness of the condition (33) for all $i \in \mathrm{Even}[\mathbb{N} \cup \{0\}]$ when the equalities (44) are satisfied. Since there were initially no shock waves in the rod, no shock wave propagates along the characteristics

$$x = at, \quad x = l + al_1 - at, \quad x = \gamma_-(r_2) + at, \quad x = l + al_2 - at, \quad x = \gamma_-(r_4) + at$$

etc., i. e. $x = \gamma_-(r_{2i}) + at$ and $x = l + al_{2i+1} - at$, $i = 0,1,\ldots$. It proves the physical correctness of the condition (33) for all $i \in \mathrm{Odd}[\mathbb{N}]$.

Conclusions. In the present paper we have obtained the necessary and sufficient conditions under which a unique classical solution of a mixed problem exists for the wave equation with discontinuous conditions in a curvilinear half-strip. We have constructed the solution in an implicit analytical form. We have proposed a method for constructing solutions to mixed problems for hyperbolic equations with discontinuous conditions in curvilinear domains.

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