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**V. S. Mukha**

*Belarusian State University of Informatics and Radioelectronics, Minsk, Republic of Belarus*

# ORTHOGONAL POLYNOMIAL MULTIDIMENSIONAL-MATRIX REGRESSION ANALYSIS

**Abstract.** The article is devoted to the orthogonal regression analysis, which is associated with the representation of the regression function by Fourier series by the multidimensional-matrix (mdm) orthogonal polynomials, in opposite to the (usual) regression analysis, when the regression function is approximated by the (usual) polynomial (by the degrees of the independent mdm input variable). We will also distinguish the classical regression analysis, when the scalar or might the classical vector-matrix mathematical approaches are used, and the mdm regression analysis, when the mdm variables and the mdm mathematical approach are used. In this article, the orthogonal regression analysis is developed on the base of the orthogonal polynomials and the mdm mathematical approach, so called the mdm orthogonal polynomial regression analysis. The known results from the theory of the orthogonal mdm polynomials and Fourier series of the vector argument are generalized to the case of the mdm argument and function. The analytical expressions for the coefficients of the second degree orthogonal polynomials and Fourier series for the potential studies are obtained. The general case of the approximation of the mdm function of the mdm argument by the Fourier series is realized programmatically as the single program function and its efficiency is confirmed by the computer calculations. The properties of the estimations of regression coefficients and unknown parameters are studied and their distributions when the normal distribution of the measurement's errors are obtained for the arbitrary covariance matrix of the errors of measurements and the arbitrary degree of the approximating polynomial. These results allow testing the hypothesis and building the hyper-rectangular confidence areas relating the orthogonal regression function. Theoretical results are confirmed by computer simulation.

**Keywords:** multidimensional-matrix orthogonal polynomials; multidimensional-matrix Fourier series; orthogonal multidimensional-matrix regression analysis

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**В. С. Муха**

*Белорусский государственный университет информатики и радиоэлектроники, Минск, Республика Беларусь*

# ОРТОГОНАЛЬНЫЙ ПОЛИНОМИАЛЬНЫЙ МНОГОМЕРНО-МАТРИЧНЫЙ РЕГРЕССИОННЫЙ АНАЛИЗ

**Аннотация**. Исследуется ортогональный регрессионный анализ, связанный с представлением функции регрессии рядом Фурье по многомерно-матричным ортогональным полиномам, в противоположность обычному регрессионному анализу, когда функция регрессии аппроксимируется обычными полиномами (степенями независимой входной переменной). Будем различать классический регрессионный анализ, когда используются скалярный или, возможно, классический векторно-матричный математический подходы, и многомерно-матричный регрессионный анализ, когда используются многомерно-матричные переменные и многомерно-матричный математический подход. В статье разрабатывается ортогональный регрессионный анализ на основе ортогональных полиномов и многомерно-матричного математического подхода, так называемый ортогональный многомерно-матричный полиномиальный регрессионный анализ. Известные результаты теории ортогональных многомерно-матричных полиномов и рядов Фурье векторного аргумента обобщаются на случай многомерно-матричных аргумента и функции. Получены аналитические выражения коэффициентов ортогональных полиномов и рядов Фурье до второй степени для возможных аналитических исследований. Программно реализован общий случай аппроксимации многомерно-матричной функции многомерно-матричного аргумента рядами Фурье в виде единичной программной функции, и ее эффективность подтверждена компьютерными расчетами. Изучены свойства коэффициентов регрессии и неизвестных параметров и их распределения при нормальном распределении ошибок измерений с произвольной ковариационной матрицей для произвольных степеней аппроксимирующих полиномов. Полученные результаты позволяют проверять гипотезы и строить гиперпрямоугольные доверительные интервалы, относящиеся к функции регрессии. Теоретические результаты подтверждены компьютерным моделированием.

**Ключевые слова**: многомерно-матричные ортогональные полиномы, многомерно-матричные ряды Фурье, ортогональный многомерно-матричный регрессионный анализ

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**Introduction.** The most important tool for research of the real systems and processes is the regression analysis. We will distinguish the classical regression analysis, when the scalar or might the classical vector-matrix mathematical approaches are used [1, 2], and the mdm regression analysis, when the mdm mathematical approach is used [3, 4]. The regression function is approximated in these cases by the polynomials in vector-matrix or mdm forms respectively. The approach is also possible when the regression function is approximated by Fourier series by the mdm orthogonal polynomials. Such a regression and regression analysis we will call orthogonal. In this article, we will deal with the mdm orthogonal regression analysis, i. e. with the orthogonal mdm polynomials, mdm Fourier series and the orthogonal mdm regression analysis.

The theory of the orthogonal polynomials of both one and several arguments dates back to Hermite [5]. Hermite in [5] and then Appel P. and Kampe de Feriet in [6] studied in details the properties of the so-called Hermite polynomials of one and two variables. The general theory of the orthogonal polynomials of many arguments is developed in the work [7]. As for the works [5–7], this theory is constructed as the theory of two bi-orthonormal sequences of the polynomials: basic and conjugate. In the work [8], it is proposed to choose a polynomial with the unit coefficient at the highest degree as the basic polynomial. This cited theory of the orthogonal polynomials of the several arguments uses only the scalar (classical) mathematical approach, and we will call it therefore the classical theory. The classical theory can be found also in [9, 10].

The foundations of the theory of multidimensional matrices were laid in the work [11] and developed in the work [12]. The ideas of the works [7, 8] were combined in the works [12, 13] on the basis of the mdm mathematical approach. This is how the mdm theory of orthogonal polynomials and Fourier series of the vector argument arose. In the work [14], the general theory of the orthogonal polynomials and Fourier series of the vector variable was extended to the vector spaces with the discrete weight.

In the article [15], the theory of orthogonal mdm polynomials and Fourier series is developed in the direction of its practical use. Contrary to the works [12, 13], the case of mdm functions is considered, and the analytical expressions for the coefficients of the second degree orthogonal polynomials and Fourier series of the vector argument are obtained.

In the current article, the theory of orthogonal mdm polynomials and Fourier series is generalized to the case of mdm functions and mdm arguments. The constraints of the work [4] such as the covariance matrix of the errors of measurements proportional to the identity matrix and the linear regression function with respect to the input mdm variable are removed.

Since the theory is created on the basis of mdm mathematical approach, the mdm notation is used in the current article. The basic definitions of the theory of multidimensional matrices in English can be found in the Appendix to the article [16] and in the article [15].

**1. Orthogonal polynomials and Fourier series of the multidimensional-matrix variable.**

***1.1. Orthogonal polynomials of the multidimensional-matrix variable.*** Let $\Omega$ be some closed region of the space $R^n$, $\rho(x)$, $x \in \Omega$, be nonnegative function (weight function) such that the integrals (the moments of the weight function $\rho(x)$)

$$\nu_{x^i} = \int_\Omega x^i \rho(x) dx < \infty, \quad i = 0,1,2,...,$$

exist, and $L_2(\rho,\Omega)$ be the space of the functions with integrable square in $\Omega$ with the weight $\rho(x)$. Here $x$ is the $q$-dimensional matrix of the size $n_1 \times n_2 \times \cdots \times n_q$, $n = n_1 n_2 \cdots n_q$, $x^i$ is (0,0)-rolled degree of the $q$-dimensional matrix $x$: $x^i = {}^{0,0}(x^i) = {}^{0,0}(x \cdot x \cdots x)$ [12, 16]. We will denote the argument $x$ as $x = (x_{j_1, j_2,...,j_q})$, $j_\alpha = 1,2,...,n_\alpha$, $\alpha = 1,2,...,q$, or as $x = (x_{\overline{j}})$, where $\overline{j} = (j_1, j_2,..., j_q)$ is the $q$-multiindex. It is supposed that the values of the $q$-multiindex $\overline{j} = (j_1, j_2,..., j_q)$ are ordered in some way and take the values $1,2,...,n$ with $n = n_1 n_2 \cdots n_q$. We will call the numbers $(n_1, n_2,\ldots, n_q)$ as the size of the matrix $x$, or as the size of the $q$-multiindex $\overline{j} = (j_1, j_2, ..., j_q)$. The number $n = n_1 n_2 \cdots n_q$ we will call the length of the $q$-dimensional matrix $x$ or the length of the $q$-multiindex $\overline{j} = (j_1, j_2, ..., j_q)$.

The theory of orthogonal polynomials of the mdm variable is created as the theory of two bi-orthonormal sequences of polynomials.

A mdm $r$ degree polynomial $Q_r(x)$, $r = 0,1,2,...$, of the $q$-dimensional-matrix variable $x \in \Omega$ is defined as follows [11, 12]:

$$Q_r(x) = \sum_{k=0}^{r} {}^{0,kq}\left(C^*_{(r,kq)} x^k\right) = \sum_{k=0}^{r} {}^{0,kq}\left(x^k C^*_{(kq,r)}\right), \tag{1}$$

where $C^*_{(r,kq)}$ are the $(r + k)q$-dimensional matrices of the coefficients,

$$C^*_{(r,kq)} = \left(c^*_{\overline{i}_1,...,\overline{i}_r,\overline{j}_1,...,\overline{j}_k}\right), \quad r = 0,1,2,..., \quad k = 0,1,2,...,r,$$

is symmetrical with respect to the $q$-multiindices $\overline{i}_1,...,\overline{i}_r$ when $r \geq 2$, symmetrical with respect to the $q$-multiindices $\overline{j}_1,...,\overline{j}_k$ when $k \geq 2$ and satisfying the conditions

$$C^*_{(r,kq)} = \left(C^*_{(kq,r)}\right)^{H(r+k)q,kq}, \quad C^*_{(kq,r)} = \left(C^*_{(r,kq)}\right)^{B(k+r)q,kq}.$$

The notations $H_{(r+k)q,kq}$ and $B_{(r+k)q,kq}$ mean the transpose substitutions of the types “back” and “forward” respectively [12, 16]. Each of the multiindices $\overline{i}_1,...,\overline{i}_r$, $\overline{j}_1,...,\overline{j}_k$ takes the values $1,2,...,n$ with $n = n_1 n_2 \cdots n_q$.

In the notations $C^*_{(r,kq)}$ and $C^*_{(kq,r)}$, the index $r$ means the degree of the polynomial, and the index $kq$ means that it is the coefficient for $x^k$.

D e f i n i t i o n 1. The sequence of the mdm polynomials $Q_r(x)$ (1) is called orthogonal in $L_2(\rho,\Omega)$ if the following conditions are satisfied:

$$\int_\Omega Q_r(x) Q_k(x) \rho(x) dx \begin{cases} = 0, & k = 0,1,...,r-1, \\ \neq 0, & k = r. \end{cases} \tag{2}$$

Two sequences of orthogonal polynomials of the mdm variable are considered: the basic sequence $P_r(x)$ and the sequence $Q_r(x)$ conjugate of $P_r(x)$, $r = 0,1,2,...$ .

D e f i n i t i o n 2. The mdm $r$ degree polynomial in $L_2(\rho,\Omega)$, $r = 0,1,2,...$, of the following form

$$P_r(x) = \sum_{k=0}^{r-1} {}^{0,kq}\left(C_{(r,kq)} x^k\right) + x^r = \sum_{k=0}^{r-1} {}^{0,kq}\left(x^k C_{(kq,r)}\right) + x^r \tag{3}$$

is called the basic polynomial, where $C_{(r,kq)}$ are $(r + k)q$-dimensional matrices of the coefficients,

$$C_{(r,kq)} = (c_{\overline{i}_1,...,\overline{i}_r,\overline{j}_1,...,\overline{j}_k}), \quad r = 0,1,2,..., \quad k = 0,1,2,...,r-1,$$

is symmetrical with respect to the $q$-multiindices $\overline{i}_1,...,\overline{i}_r$ when $r \geq 2$, symmetrical with respect to the $q$-multiindices $\overline{j}_1,...,\overline{j}_k$ when $k \geq 2$ and satisfying the conditions

$$C_{(r,kq)} = C^{H(r+k)q,kq}_{(kq,r)}, \quad C_{(kq,r)} = C^{B(k+r)q,kq}_{(r,kq)}.$$

D e f i n i t i o n 3. The mdm polynomial $P_r(x)$ (3) is called the basic orthogonal $r$ degree polynomial in $L_2(\rho,\Omega)$ if it is orthogonal to the homogeneous polynomials $1, x, x^2, ..., x^{r-1}$, $x^k = {}^{0,0}(x^k)$:

$$\int_\Omega P_r(x) x^k \rho(x) dx \begin{cases} = 0, & k = 0,1,...,r-1, \\ \neq 0, & k = r. \end{cases} \tag{4}$$

D e f i n i t i o n 4. The sequences of the mdm polynomials $P_r(x)$ (3) and $Q_r(x)$ (1) are called completely orthonormal in $L_2(\rho,\Omega)$ if the conditions (2), (4) are satisfied and the following condition:

$$\int_\Omega Q_r(x) P_k(x) \rho(x) dx = \begin{cases} 0, & k \neq r, \\ r! E^{(s)}(0, rq), & k = r, \end{cases} \tag{5}$$

is satisfied too. Here $E^{(s)}(0,rq)$ is the symmetrical (0,*rq*)-identity matrix [12]. The matrix $E^{(s)}(0,rq)$ is the 2*rq*-dimensional matrix which has such a property that for any *mq*-dimensional matrix $C=(c_{\bar{i}_1,\dots,\bar{i}_{m-r},\bar{j}_1,\dots,\bar{j}_r})$ with $m\geq r$ symmetrical with respect to the multiindices $\bar{j}_1,\bar{j}_2,\dots,\bar{j}_r$ the following equality is fulfilled [12]: ${}^{0,rq}(CE^{(s)}(0,rq))=C$.

R e m a r k. This definition is new compared to the author's previous works. In this definition, the notion of the symmetrical (0,*rq*)-identity matrix $E^{(s)}(0,rq)$ is used. This made it possible to generalize the theory to the mdm argument.

Let us introduce the initial *i*-th order moments $\nu_{x^i}$ and the initial-central and central-initial $(i+j)$-th order moments $\nu_{x^i x_c^j}$, $\nu_{x_c^i x^j}$ of the weight function $\rho(x)$:

$$\nu_{x^0}=\int_\Omega \rho(x)dx,$$

$$\nu_{x^i x^j}=\nu_{x^{i+j}}=\int_\Omega x^{i+j}\rho(x)dx,\quad i+j=1,2,\dots, \tag{6}$$

$$\nu_{x^i x_c^j}=\int_\Omega x^i(x^j-\nu_{x^j})\rho(x)dx,\quad i+j=1,2,\dots, \tag{7}$$

$$\nu_{x_c^i x^j}=\int_\Omega (x^i-\nu_{x^i})x^j\rho(x)dx,\quad i+j=1,2,\dots, \tag{8}$$

where $x^i$, $x^i(x^j-\nu_{x^j})$, $(x^i-\nu_{x^i})x^j$ are (0,0)-rolled degrees and (0,0)-rolled products of the matrices [12]. We will often avoid the notation (0,0)-rolled product and will write *yx* instead of ${}^{0,0}(yx)$.

The moments (7), (8) have the following properties:

$$\nu_{x^i x_c^j}=\nu_{x_c^i x^j}=\nu_{x_c^i x_c^j}=\nu_{x^{i+j}}-\nu_{x^i}\nu_{x^j}, \tag{9}$$

$$\nu_{x_c^i x_c^j}=(\nu_{x_c^j x_c^i})^{B_{iq+jq,iq}}=\nu_{x^{j+i}}-\nu_{x^j}\nu_{x^i},$$

where $B_{iq+jq,iq}$ is the transpose substitution of the type "forward" [12]. These properties are proved by calculation of the formulae (7), (8). The properties (9) allow us to use the following notations:

$$\mu_{x^i x^j}=\nu_{x^{i+j}}-\nu_{x^i}\nu_{x^j},\quad \mu_{x^j x^i}=\nu_{x^{j+i}}-\nu_{x^j}\nu_{x^i},\quad \mu_{x^i x^j}=(\mu_{x^j x^i})^{B_{iq+jq,iq}}.$$

Let us introduce also the mutual moments

$$\nu_{y^i x^j}=\int_\Omega y^i(x)x^j\rho(x)dx \tag{10}$$

with the properties

$$\nu_{y^i x_c^j}=\nu_{y_c^i x^j}=\nu_{y_c^i x_c^j}=\int_\Omega y^i(x)(x^j-\nu_{x^j})\rho(x)dx=\nu_{y^i x^j}-\nu_{y^i}\nu_{x^j}=\mu_{y^i x^j}.$$

The weight function $\rho(x)$ in the case $\nu_{x^0}=1$ represents the probability density function of some random *q*-dimensional matrix $\xi$.

T h e o r e m 1 (generalization of the theorem [12]). *If the sequences of the mdm polynomials* $P_r(x)$ (3) *and* $Q_r(x)$ (1) *are completely orthonormal in* $L_2(\rho,\Omega)$, *i. e. they satisfy the conditions* (2), (4), (5), *then the coefficients* $C_{(r,kq)}$ *of the basic r degree polynomial* $P_r(x)$ (3), $r=0,1,\dots,$ *are defined by the following mdm system of the linear algebraic equations*

$$\nu_{x^r x^i}+\sum_{k=0}^{r-1}{}^{0,kq}(C_{(r,kq)}\nu_{x^k x^i})=0,\quad i=0,1,\dots,r-1,$$

*where i is the number of the equation, and the coefficients* $C^*_{(r,kq)}$ *of the conjugate r degree polynomial* $Q_r(x)$ (1) *are defined by the expression*

$$C^*_{(r,kq)}=r!\,{}^{0,rq}\left({}^{0,rq}B^{-1}_{(r,r)}C_{(r,kq)}\right),$$

*where* ${}^{0,rq}B^{-1}_{(r,r)}$ *is the matrix* $(0,r)$*-inverse to the following matrix* $B_{(r,r)}$*:*

$$B_{(r,r)} = \nu_{x^r x^r} + \sum_{k=0}^{r-1} {}^{0,kq}\left(C_{(r,kq)}\nu_{x^k x^r}\right) + \sum_{k=0}^{r-1} {}^{0,kq}\left(\nu_{x^r x^k} C_{(kq,r)}\right) + \sum_{k=0}^{r-1}\sum_{s=0}^{r-1} {}^{0,kq}\left(C_{(r,kq)} {}^{0,sq}(\nu_{x^k x^s} C_{(sq,r)})\right).$$

*The coefficients* $C_{(kq,r)}$ *of the basic r degree polynomial* $P_r(x)$ (3)*,* $r = 0,1,...,$ *are defined by the following mdm system of the linear algebraic equations:*

$$\nu_{x^i x^r} + \sum_{k=0}^{r-1} {}^{0,kq}\left(\nu_{x^i x^k} C_{(kq,r)}\right) = 0, \quad i = 0,1,...,r-1,$$

*where i is the number of the equation, and the coefficients* $C^*_{(kq,r)}$ *of the conjugate r degree polynomial* $Q_r(x)$ (1) *are defined by the expression*

$$C^*_{(kq,r)} = r!\, {}^{0,rq}\left(C_{(kq,r)} {}^{0,rq}B^{-1}_{(r,r)}\right).$$

***1.2. Fourier series by the orthogonal polynomials.*** Fourier series for the $p$-dimensional-matrix function $y(x)$ of the $q$-dimensional-matrix variable $x \in \Omega \subseteq R^n$ by the conjugate orthogonal polynomials $Q_r(x)$ (1) has the following form:

$$y(x) \sim \sum_{r=0}^{\infty} \frac{1}{r!} {}^{0,rq}\left(\beta_{(p,rq)} Q_r(x)\right) = \sum_{r=0}^{\infty} \frac{1}{r!} {}^{0,rq}\left(Q_r(x)\beta_{(rq,p)}\right), \tag{11}$$

where $\beta_{(p,rq)} = (\beta_{j_1,...,j_p,\bar{i}_1,...,\bar{i}_r})$ are $(p + rq)$-dimensional symmetrical when $r \geq 2$ with respect to the multiindices $\bar{i}_1,...,\bar{i}_r$ matrices of the $r$ degree of the coefficients, $\beta_{(rq,p)} = \left(\beta_{(p,rq)}\right)^{B_{p+rq,rq}}$. They are defined by the expressions [12]

$$\beta_{(p,rq)} = \int_{\Omega} {}^{0,0}\left(y(x)P_r(x)\right)\rho(x)dx. \tag{12}$$

Substitution of the polynomial $P_r(x)$ (3) into (12) gives the following expression for the coefficients $\beta_{(p,rq)}$:

$$\beta_{(p,rq)} = \int_{\Omega} {}^{0,0}\left(y(x)P_r(x)\right)\rho(x)dx = \int_{\Omega} {}^{0,0}\left(y(x)\left(x^r + \sum_{k=0}^{r-1} {}^{0,kq}\left(x^k C_{(kq,r)}\right)\right)\right)\rho(x)dx =$$

$$= E\left({}^{0,0}(yx^r) + \sum_{k=0}^{r-1} {}^{0,kq}\left({}^{0,0}(yx^k)C_{(kq,r)}\right)\right) = \nu_{yx^r} + \sum_{k=0}^{r-1} {}^{0,kq}\left(\nu_{yx^k} C_{(kq,r)}\right), \quad r = 0,1,2,... .$$

Fourier series by the basic orthogonal polynomials $P_r(x)$ (3) is obtained analogously:

$$y(x) \sim \sum_{r=0}^{\infty} \frac{1}{r!} {}^{0,rq}\left(\alpha_{(p,rq)} P_r(x)\right) = \sum_{r=0}^{\infty} \frac{1}{r!} {}^{0,rq}\left(P_r(x)\alpha_{(rq,p)}\right), \tag{13}$$

where

$$\alpha_{(p,rq)} = \int_{\Omega} {}^{0,0}\left(y(x)Q_r(x)\right)\rho(x)dx. \tag{14}$$

Since

$$Q_r(x) = r!\, {}^{0,rq}\left(P_r(x) {}^{0,r}B^{-1}_{(r,r)}\right),$$

then

$$\alpha_{(p,rq)} = \int_{\Omega} {}^{0,0}\left(y(x)Q_r(x)\right)\rho(x)dx = r!\int_{\Omega} {}^{0,0}\left(y(x) {}^{0,rq}(P_r(x) {}^{0,rq}B^{-1}_{(r,r)})\right)\rho(x)dx =$$

$$= r!\int_{\Omega} {}^{0,rq}\left({}^{0,0}\left(y(x)P_r(x)\right) {}^{0,rq}B^{-1}_{(r,r)}\right)\rho(x)dx = r!\, {}^{0,rq}\left(\beta_{(p,rq)} {}^{0,rq}B^{-1}_{(r,r)}\right), \quad r = 0,1,2,... .$$

Approximation of the scalar (zero-dimensional-matrix, $p = 0$) function $y(\xi)$ of the random $q$-dimensional matrix $\xi$ with the probability density function $\rho(x)$ by the finite sum of the Fourier series:

$$s_m(\xi)=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\beta_{(p,rq)}Q_r(\xi)\right)=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\alpha_{(p,rq)}P_r(\xi)\right)$$

provides the minimum of the mean square error (m.s.e.) of the approximation

$$r_m^2=E\left({}^{0,0}\left(y(\xi)-s_m(\xi)\right)^2\right)=\int_{\Omega}{}^{0,0}\left(y(x)-s_m(x)\right)^2\rho(x)dx.$$

The minimal value $r_{m\,\min}^2$ of the m.s.e. is defined by the expression [12]

$$r_{m\,\min}^2=E\left(y^2(\xi)\right)-\sum_{r=0}^{m}\frac{1}{r!}{}^{0,r}\left(\beta_{(p,rq)}\alpha_{(p,rq)}\right),$$

where $E(\cdot)$ means the mathematical expectation by the weight function $\rho(x)$.

***1.3. The polynomials orthogonal with the discrete weight function.*** The theory of the polynomials orthogonal with the continuous weight function $\rho(x)$ outlined above coincides with the theory of the polynomials orthogonal with the discrete weight function $(p_k, x_k)$, when the $l$ distinct points are given in the region $\Omega \subseteq R^n$ with positive weights $p_1, p_2, ..., p_l$ and the measure $\mu$ of the region $\Omega$ is defined by the formula $\mu(\Omega)=\sum_{x_k\in\Omega}p_k$ [14]. One talks in this case about the polynomials orthogonal on the system of the points. The moments (6) are defined in this case by the expression:

$$\nu_{x^i x^j}=\nu_{x^{i+j}}=\int_{\Omega}x^{i+j}d\mu=\sum_{k=1}^{l}x_k^{i+j}p_k,\quad i+j=1,2,...,$$

and the mutual moments (10) are defined by the expression

$$\nu_{y^i x^j}=\int_{\Omega}y^i(x)x^j d\mu=\sum_{k=1}^{l}y_k^i x_k^j p_k,\quad i+j=1,2,...,$$

where $y_k = y(x_k),\ k = 1,2,...,l.$

We will call the discrete weight function with $\nu_{x^0}=1$ as the discrete distribution of some random variable $\xi$. The important discrete distribution is so called empirical, or sample distribution, when $x_i$ are the sample values of a random variable $\xi$, and $p_i = 1/l$, where $l$ is the length of the sample. If the empirical distribution is used, then the approximation (11), (13) is called empirical.

The orthogonality conditions (2), (4), (5) of the polynomials for the sample distribution as the weight function look like:

$$\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)P_k(x_i)\right)=\begin{cases}0, & k\neq r,\\ \neq 0, & k=r,\end{cases}\tag{15}$$

$$\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)Q_k(x_i)\right)=\begin{cases}0, & k\neq r,\\ \neq 0, & k=r,\end{cases}$$

$$\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)P_k(x_i)\right)=\begin{cases}0, & k\neq r,\\ r!E^{(s)}(0,rq), & k=r.\end{cases}\tag{16}$$

In this article, we will deal with the polynomials orthogonal on the system of the points, and, therefore, with the empirical approximation.

***1.4. The multidimensional-matrix polynomial approximation by Fourier approximation.*** It is of interest to obtain the coefficients $c_{(p,kq)}$ of the approximation of $p$-dimensional-matrix function $y(x)$ by the mdm $m$ degree polynomial:

$$y(x)\sim\sum_{k=0}^{m}{}^{0,kq}\left(c_{(p,kq)}x^k\right)\tag{17}$$

in the case when Fourier approximation by the basic orthogonal polynomials $P_k(x)$ (13) the same degree of this function is obtained

$$y(x)\sim\sum_{k=0}^{m}\frac{1}{k!}\,{}^{0,kq}\left(\alpha_{(p,kq)}P_k(x)\right). \tag{18}$$

We will call the approximation (17) as the representation of the function $y(x)$ by the degrees of independent variable $x$, and the approximation (18) as the representation of the function $y(x)$ by orthogonal polynomials.

The polynomial $P_k(x)$ of the fixed degree $k$ provides in the expression (18) the following summand:

$$\frac{1}{k!}\,{}^{0,kq}\left(\alpha_{(p,kq)}P_k(x)\right)=\frac{1}{k!}\,{}^{0,kq}\left(\alpha_{(p,kq)}\left(\sum_{i=0}^{k}{}^{0,iq}\left(C_{(k,iq)}x^i\right)\right)\right)=\sum_{i=0}^{k}\frac{1}{k!}\,{}^{0,iq}\left({}^{0,kq}(\alpha_{(p,kq)}C_{(k,iq)})x^i\right). \tag{19}$$

The variable $x$ of the degree $r$, $r \le k$, appears in the expression (19) in the summand ${}^{0,rq}\left({}^{0,kq}(\alpha_{(p,kq)}C_{(k,rq)})x^r\right)/k!$. Summation of the coefficients at $x^r$ by $k$ from $r$ to $m$ gives the following formula for the desired coefficients:

$$c_{(p,rq)}=\sum_{k=r}^{m}\frac{1}{k!}\,{}^{0,kq}\left(\alpha_{(p,kq)}C_{(k,rq)}\right),\quad r=0,1,2,..,m. \tag{20}$$

If one takes in account that $C_{(i,i)}=E^{(s)}(0,iq)$ is the symmetrical identity matrix which ensures the equality ${}^{0,iq}(C_iC_{(i,i)})=\alpha_{(p,iq)}$, then instead of (20) we will have the expression

$$c_{(p,rq)}=\frac{1}{r!}\alpha_{(p,rq)}+\sum_{k=r+1}^{m}\frac{1}{k!}\,{}^{0,kq}(\alpha_{(p,kq)}C_{(k,rq)}),\quad r=0,1,2,..,m. \tag{21}$$

Let Fourier approximation of the function $y(x)$ be obtained by the conjugate orthogonal polynomials $Q_k(x)$:

$$y(x)\sim\sum_{k=0}^{m}\frac{1}{k!}\,{}^{0,kq}\left(\beta_{(p,kq)}Q_k(x)\right). \tag{22}$$

Then the following expression for the coefficients $c_{(p,rq)}$ in (17) could be obtained:

$$c_{(p,rq)}=\sum_{k=r}^{m}\frac{1}{k!}\,{}^{0,kq}\left(\beta_{(p,kq)}C^{*}_{(k,rq)}\right),\quad r=0,1,2,..,m. \tag{23}$$

The algorithm of the approximation of the functions $y(x)$ by Fourier series (18) and (22) was realized programmatically in the form of a standard Matlab function for general case of the theorem 1 and was checked on many functions. The considered approximations have undoubted advantages compared to the classical approach: algorithmical generality and extensive possibilities. However, they have certain hardware limitations: out of memory and unacceptably long calculation time for the personal computer in the case of big data.

***1.5. The orthogonal 0–2 degrees polynomials and Fourier series.*** In the work [12], the expressions of zero and first degree orthonormal polynomials and the particular cases of the second degree polynomials are obtained. These results are completed in this article by the general expressions of the second degree polynomials and Fourier series. The complete expressions are presented in the Table for the case $\nu_{x^0}=1$.

**Orthogonal polynomials and Fourier series up to second degree inclusive**

| Polynomials $P(x)$ | Polynomials $Q(x)$ |
|---|---|
| $P_0(x)=1$ | $Q_0(x)=1$ |
| $P_1(x)=C_{(1,0)}+x,\quad C_{(1,0)}=-\nu_x$ | $Q_1(x)={}^{0,q}\left({}^{0,q}B^{-1}_{(1,1)}P_1(x)\right),\quad B_{(1,1)}=\mu_{xx}=\nu_{xx}-\nu_x\nu_x$ |

*End of table*

| Polynomials $P(x)$ | Polynomials $Q(x)$ |
|---|---|
| $P_2(x) = C_{(2,0)} + {}^{0,q}(C_{(2,1)}x) + x^2$,<br>$C_{(2,1)} = -{}^{0,q}\left(\mu_{x^2x}\,{}^{0,q}\mu_{xx}^{-1}\right)$,<br>$C_{(2,0)} = -{}^{0,q}(C_{(2,1)}\nu_x) - \nu_{x^2}$,<br>$\mu_{x^2x} = \nu_{x^3} - \nu_{x^2}\nu_x$ | $Q_2(x) = 2!^{0,2q}\left({}^{0,2q}B_{(2,2)}^{-1}P_2(x)\right)$,<br>$B_{(2,2)} = \mu_{x^2x^2} - {}^{0,q}\left({}^{0,q}\left(\mu_{x^2x}\,{}^{0,q}\mu_{xx}^{-1}\right)\mu_{xx^2}\right)$,<br>$\mu_{x^2x^2} = \nu_{x^4} - \nu_{x^2}\nu_{x^2}$,<br>$\mu_{x^2x} = \nu_{x^3} - \nu_{x^2}\nu_x$ |
| Fourier series by the polynomials $P(x)$ | Fourier series by the polynomials $Q(x)$ |
| $y(x) \sim {}^{0,0}\left(\alpha_{(p,0q)}P_0(x)\right) = \nu_y$, | $y(x) \sim {}^{0,0}\left(\beta_{(p,0q)}Q_0(x)\right) = \nu_y$ |
| $y(x) \sim {}^{0,0}\left(\alpha_{(p,0q)}P_0(x)\right) + {}^{0,q}\left(\alpha_{(p,q)}P_1(x)\right)$,<br>$\alpha_{(p,0q)} = \nu_y$, $\alpha_{(p,q)} = {}^{0,q}\left(\mu_{yx}\,{}^{0,q}\mu_{xx}^{-1}\right)$ | $y(x) \sim {}^{0,0}\left(\beta_{(p,0q)}Q_0(x)\right) + {}^{0,q}\left(\beta_{(p,q)}Q_1(x)\right)$,<br>$\beta_{(p,0q)} = \nu_y$, $\beta_{(p,q)} = \mu_{yx}$ |
| $y(x) \sim {}^{0,0}\left(\alpha_{(p,0q)}P_0(x)\right) + {}^{0,q}\left(\alpha_{(p,q)}P_1(x)\right) +$<br>$+\frac{1}{2}\,{}^{0,2q}\left(\alpha_{(p,2q)}P_2(x)\right)$,<br>$\alpha_{(p,0q)} = {}^{0,0}\left(\beta_{(p,0q)}\,{}^{0,0}B_{(0,0)}^{-1}\right) = \nu_y$,<br>$\alpha_{(p,q)} = {}^{0,q}\left(\beta_{(p,q)}\,{}^{0,q}B_{(1,1)}^{-1}\right) = {}^{0,q}(\mu_{yx}\,{}^{0,q}\mu_{xx}^{-1})$,<br>$\alpha_{(p,2q)} = 2^{0,2q}\left(\beta_{(p,2q)}\,{}^{0,2q}B_{(2,2)}^{-1}\right)$ | $y(x) \sim {}^{0,0}\left(\beta_{(p,0q)}Q_0(x)\right) + {}^{0,q}\left(\beta_{(p,q)}Q_1(x)\right) +$<br>$+\frac{1}{2}\,{}^{0,q}\left(\beta_{(p,2q)}Q_2(x)\right)$,<br>$\beta_{(p,0q)} = \nu_y$,<br>$\beta_{(p,q)} = \mu_{yx}$,<br>$\beta_{(p,2q)} = \mu_{yx^2} - {}^{0,q}\left(\mu_{yx}\,{}^{0,q}\left({}^{0,q}\mu_{xx}^{-1}\mu_{x^2x}\right)\right)$ |

Coefficients of the approximation of the function $y(x)$ by the series (17) by the degrees of the variable $x$ up to the second degree inclusive ($m = 2$) are defined by the following expressions defined by the formula (21):

$$c_{(p,0)} = \frac{1}{0!}\alpha_{(p,0q)} + \sum_{k=1}^{2}\frac{1}{k!}\,{}^{0,kq}(\alpha_{(p,kq)}C_{(kq,0)}) = \alpha_{(p,0q)} + {}^{0,q}(\alpha_{(p,q)}C_{(q,0)}) + \frac{1}{2!}\,{}^{0,2q}(\alpha_{(p,2q)}C_{(2q,0)}),$$

$$c_{(p,q)} = \frac{1}{1!}\alpha_{(p,q)} + \sum_{k=2}^{2}\frac{1}{k!}\,{}^{0,kq}(\alpha_{(p,kq)}C_{(kq,1)}) = \alpha_{(p,q)} + \frac{1}{2!}\,{}^{0,2q}(\alpha_{(p,2q)}C_{(2q,1)}),$$

$$c_{(p,2q)} = \frac{1}{2!}\alpha_{(p,2q)}.$$

**2. Orthogonal regression analysis.**

***2.1. Orthogonal regression analysis. Statement of the problem.*** Let the $p$-dimensional-matrix function $y(x)$ of the $q$-dimensional-matrix variable $x$ be represented by the conjugate orthogonal polynomials $Q_r(x)$ and is measured with errors $\varepsilon(x)$ in the points $x_1, x_2, \ldots, x_l$ so that

$$y_{o,i} = y_i + \varepsilon_i = \sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\beta_{(p,rq)}Q_r(x_i)\right) + \varepsilon_i, \quad i = 1, 2, \ldots, l, \tag{24}$$

and be represented by the degrees of the variable $x$ (what is the same as (24)),

$$y_{o,i} = y_i + \varepsilon_i = \sum_{r=0}^{m}{}^{0,rq}\left(c_{(p,rq)}x^r\right) + \varepsilon_i, \quad i = 1, 2, \ldots, l, \tag{25}$$

where $\varepsilon_i$ are the values of the random $p$-dimensional matrix of the errors $\varepsilon(x_i)$, and

$$y_i = \sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\beta_{(p,rq)}Q_r(x_i)\right), \tag{26}$$

or, what is the same,

$$y_i = \sum_{r=0}^{m} {}^{0,rq}\left(c_{(p,rq)} x_i^r\right), \tag{27}$$

is the regression function. We will suppose that $E(\varepsilon_i) = 0$ is the mean value of $\varepsilon_i$, $V_\varepsilon = E({}^{0,0}\varepsilon_i^2) = E({}^{0,0}\varepsilon^2)$ is the covariance matrix of the errors $\varepsilon_i$, $E(\cdot)$ means the mathematical expectation, $\varepsilon_i$ and $\varepsilon_j$ are independent random matrices by $i$, so that

$$E\left({}^{0,0}(\varepsilon_i \varepsilon_j)\right) = \begin{cases} 0, & i \neq j, \\ V_\varepsilon, & i = j. \end{cases}$$

We suppose also, that the covariance matrix $V_\varepsilon$ of the errors $\varepsilon_i$ is unknown.

The measurements $(y_{o,i}, x_i)$, $i = 1,2,...,l,$ allow us to approximate the function $y(x)$ by the empirical Fourier series by the orthogonal polynomials

$$\widehat{y}_i = (\widehat{y}_{\overline{i}(p)}) = \sum_{r=0}^{m} \frac{1}{r!} {}^{0,rq}\left(\widehat{\beta}_{(p,rq)} Q_r(x_i)\right), \quad i = 1,2,...,l, \quad \overline{i}_{(p)} = i_1, i_2, ..., i_p, \tag{28}$$

or, what is the same, by the degrees of the argument $x$,

$$\widehat{y}_i = (\widehat{y}_{\overline{i}(p)}) = \sum_{r=0}^{m} {}^{0,rq}(\widehat{c}_{(p,rq)} x_i^r), \quad i = 1,2,...,l, \quad \overline{i}_{(p)} = i_1, i_2, ..., i_p, \tag{29}$$

where the coefficients $\widehat{\beta}_{p,rq}$ are defined by the following expression:

$$\widehat{\beta}_{(p,rq)} = (\widehat{\beta}_{\overline{i}(p),\overline{\overline{j}}(r)}) = \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(y_{o,i} P_r(x_i)\right), \ \overline{i}_{(p)} = i_1, i_2, ..., i_p, \ \overline{\overline{j}}_{(r)} = \overline{j}_{q,1}, \overline{j}_{q,2}, ..., \overline{j}_{q,r}, \ r = 0,1,2,...,m, \tag{30}$$

which follows from the expression (12). The coefficients $\widehat{\beta}_{(p,rq)}$ can be considered as the estimations of the regression coefficients $\beta_{(p,rq)}$ in (26), and $\widehat{y}_i$ in (28), (29) can be considered as the estimation of the hypothetical regression function (26) (estimation of the response).

The estimations $\widehat{c}_{(p,kq)}$ of the coefficients $c_{(p,kq)}$ in representation by the degrees of the variable $x$ (25), (27), (29) are defined by the following expressions

$$\widehat{c}_{(p,rq)} = (\widehat{c}_{i(p),\overline{\overline{j}}(rq)}) = \sum_{k=r}^{m} \frac{1}{k!} {}^{0,kq}(\widehat{\beta}_{(p,kq)} C^*_{(q,rq)}), \quad r = 0,1,2,..,m, \tag{31}$$

which follows from the expression (23).

The $p$-dimensional-matrix function $y(x)$ of the $q$-dimensional-matrix variable $x$ can be represented also by the basic orthogonal polynomials $P_r(x)$ and can be measured with errors $\varepsilon(x)$ in the points $x_1, x_2, ..., x_l$ so that

$$y_{o,i} = y_i + \varepsilon_i = \sum_{r=0}^{m} \frac{1}{r!} {}^{0,rq}\left(\alpha_{(p,rq)} P_r(x_i)\right) + \varepsilon_i, \quad i = 1,2,...,l. \tag{32}$$

In this case, the estimations $\widehat{\alpha}_{(p,rq)}$ of the coefficients $\alpha_{(p,rq)}$ are defined by the expressions

$$\widehat{\alpha}_{(p,rq)} = (\widehat{\alpha}_{\overline{i}(p),\overline{\overline{j}}(r)}) = \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(y_{o,i} Q_r(x_i)\right), \ \overline{i}_{(p)} = i_1, i_2, ..., i_p, \ \overline{\overline{j}}_{(r)} = \overline{j}_{q,1}, \overline{j}_{q,2}, ..., \overline{j}_{q,r}, \ r = 0,1,2,...,m, \tag{33}$$

which follows from the expression (14), and the estimations $\widehat{c}_{(p,kq)}$ of the coefficients $c_{(p,kq)}$ in the representation by the degrees of the variable $x$ (25), (27), (29) are defined by the following expressions:

$$\widehat{c}_{(p,rq)} = (\widehat{c}_{i(p),\overline{\overline{j}}(rq)}) = \sum_{k=r}^{m} \frac{1}{k!} {}^{0,kq}(\widehat{\alpha}_{(p,kq)} C_{(k,rq)}), \quad r = 0,1,2,..,m, \tag{34}$$

which follows from the expression (20). We have now the following regression function:

$$\widehat{y}_i = (\widehat{y}_{\overline{i}(p)}) = \sum_{r=0}^{m} \frac{1}{r!} {}^{0,rq}\left(\widehat{\alpha}_{(p,rq)} P_r(x_i)\right), \quad i = 1,2,...,l, \quad \overline{i}_{(p)} = i_1, i_2, ..., i_p. \tag{35}$$

The estimation $\widehat{V}_\varepsilon$ of the unknown $V_\varepsilon = E({}^{0,0}\varepsilon^2)$ can be obtained by the following expression:

$$\widehat{V}_\varepsilon = \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}(y_{o,i} - \widehat{y}_i)^2. \tag{36}$$

The problem consists of the study of properties of the empirical regression functions (28), (29), (35) and the estimation $\widehat{V}_\varepsilon$ (36).

Let us note that the representation of the regression function by the degrees of the argument $x$ (25) is more convenient in practical use. It is supposed that the estimations $\widehat{\beta}_{(p,rq)}$ (30) or $\widehat{\alpha}_{(p,rq)}$ (33) are calculated at first and then they are recalculated to the estimations $\widehat{c}_{(p,rq)}$ by the formulae (31) or (34). In what follows we will consider all cases.

***2.2. Properties of the estimations of the regression coefficients.*** We start from the estimations $\widehat{\beta}_{(p,rq)}$, $r = 0,1,...,m$, (30). Substitution (24) into (30) gives

$$\widehat{\beta}_{(p,rq)} = \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\left(\sum_{s=0}^{m} \frac{1}{s!}\, {}^{0,sq}\left(\beta_{(p,sq)} Q_s(x_i)\right) + \varepsilon_i\right) P_r(x_i)\right) =$$

$$= \frac{1}{l}\sum_{i=1}^{l}\sum_{s=0}^{m}\frac{1}{s!}\, {}^{0,0}\left({}^{0,sq}\left(\beta_{(p,sq)} Q_s(x_i)\right) P_r(x_i)\right) + \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\varepsilon_i P_r(x_i)\right) =$$

$$= \frac{1}{l}\sum_{i=1}^{l}\sum_{s=0}^{m}\frac{1}{s!}\, {}^{0,sq}\left(\beta_{(p,sq)}\, {}^{0,0}\left(Q_s(x_i) P_r(x_i)\right)\right) + \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\varepsilon_i P_r(x_i)\right).$$

We receive the following expression due to the orthogonality of the polynomials $Q_s(x)$ and $P_r(x)$:

$$\widehat{\beta}_{(p,rq)} = \beta_{(p,rq)} + \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\varepsilon_i P_r(x_i)\right), \tag{37}$$

and

$$E_{\widehat{\beta}(p,rq)} = E(\widehat{\beta}_{(p,rq)}) = E\left(\beta_{(p,rq)} + \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\varepsilon_i P_r(x_i)\right)\right) = \beta_{(p,rq)}, \tag{38}$$

where $E(\cdot)$ is the mathematical expectation. The expression (38) means that the estimation $\widehat{\beta}_{(p,rq)}$ is unbiased. We have from (37), (38) that

$$\widehat{\beta}_{(p,rq)} - E(\widehat{\beta}_{(p,rq)}) = \widehat{\beta}_{(p,rq)} - \beta_{(p,rq)} = \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\varepsilon_i P_r(x_i)\right) = \mathring{\widehat{\beta}}_{(p,rq)}.$$

The covariance between $\widehat{\beta}_{(p,rq)}$ and $\widehat{\beta}_{(p,sq)}$ is

$$V_{\widehat{\beta}r,s} = \operatorname{cov}(\widehat{\beta}_{(p,rq)}, \widehat{\beta}_{(p,sq)}) = E\left((\widehat{\beta}_{(p,rq)} - \beta_{(p,rq)})(\widehat{\beta}_{(p,sq)} - \beta_{(p,sq)})\right) =$$

$$= E\left(\frac{1}{l^2}\sum_{i=1}^{l}\sum_{j=1}^{l} {}^{0,0}\left(\left(\varepsilon_i P_r(x_i)\right)\left(\varepsilon_j P_s(x_j)\right)\right)\right) = (v_{\widehat{\beta},\overline{i}(p),\overline{\overline{j}}(rq),\overline{k}(p),\overline{\overline{l}}(sq)}). \tag{39}$$

Let us transform the expression ${}^{0,0}\left((\varepsilon_i P_r(x_i))(\varepsilon_j P_s(x_j))\right)$. Since

$$U = {}^{0,0}\left((\varepsilon P_r)(\varepsilon P_s)\right) = {}^{0,0}(\varepsilon_{\overline{i}(p)} p_{\overline{\overline{j}}(rq)} \varepsilon_{\overline{k}(p)} p_{\overline{\overline{l}}(sq)}) = (u_{\overline{i}(p),\overline{\overline{j}}(rq),\overline{k}(p),\overline{\overline{l}}(sq)}) = (p_{\overline{\overline{j}}(rq)} \varepsilon_{\overline{i}(p)} \varepsilon_{\overline{k}(p)} p_{\overline{\overline{l}}(sq)}) =$$

$$= {}^{0,0}(P_r \varepsilon\varepsilon P_s) = Z = (z_{\overline{\overline{j}}(rq),\overline{i}(p),\overline{k}(p),\overline{\overline{l}}(sq)}),$$

where $\overline{i}_{(p)}$, $\overline{k}_{(p)}$ are the $p$-multiindices; $\overline{\overline{j}}_{(rq)}$, $\overline{\overline{l}}_{(sq)}$ are the sets of the multiindices, consisting of $r$ and $s$ $q$-multiindices respectively, then $u_{\overline{i}(p),\overline{\overline{j}}(rq),\overline{k}(p),\overline{\overline{l}}(sq)} = z_{\overline{\overline{j}}(rq),\overline{i}(p),\overline{k}(p),\overline{\overline{l}}(sq)}$. This means, that the matrices $U$ and $Z$ are connected to each other by the transpose operation:

$$U = Z^{T(r,s)},$$

where

$$T(r,s)=\begin{pmatrix}\bar{i}_{(p)},\bar{\bar{j}}_{(rq)},\bar{k}_{(p)},\bar{\bar{l}}_{(sq)}\\ \bar{\bar{j}}_{(rq)},\bar{i}_{(p)},\bar{k}_{(p)},\bar{\bar{l}}_{(sq)}\end{pmatrix}=(B_{p+rq,p},E_{p+sq}), \tag{40}$$

and $B_{p+rq,p}$ is the transpose substitution on $p+rq$ indices of the type “forward” on $p$ indices; $E_{p+sq}$ is the identical substitution on $(p+sq)$ indices. The expression (39) has now the following form:

$$V_{\hat{\beta}_{r,s}}=(v_{\beta,\bar{i}_{(p)},\bar{\bar{j}}_{(rq)},\bar{k}_{(p)},\bar{\bar{l}}_{(sq)}})=\frac{1}{l^2}\sum_{i=1}^{l}\sum_{j=1}^{l}{}^{0,0}\left(P_r(x_i)E(\varepsilon_i\varepsilon_j)P_s(x_j)\right)^{T(r,s)}. \tag{41}$$

The expression (41) is simplified due to the independence of $\varepsilon_i$ and $\varepsilon_j$ when $i\neq j$:

$$V_{\hat{\beta}_{r,s}}=(v_{\beta,\bar{i}_{(p)},\bar{\bar{j}}_{(rq)},\bar{k}_{(p)},\bar{\bar{l}}_{(sq)}})=\frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)E(\varepsilon_i\varepsilon_i)P_s(x_i)\right)^{T(r,s)}=\frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)V_\varepsilon P_s(x_i)\right)^{T(r,s)}. \tag{42}$$

The expression (42) can be represented in other form. If we transform the expression $U={}^{0,0}\left(\left(\varepsilon_iP_r(x_i)\right)\left(\varepsilon_jP_s(x_j)\right)\right)$ in (39) as follows

$$U={}^{0,0}\left((\varepsilon P_r)(\varepsilon P_s)\right)={}^{0,0}(\varepsilon_{\bar{i}_{(p)}}p_{\bar{\bar{j}}_{(rq)}}\varepsilon_{\bar{k}_{(p)}}p_{\bar{\bar{l}}_{(sq)}})=(u_{\bar{i}_{(p)},\bar{\bar{j}}_{(rq)},\bar{k}_{(p)},\bar{\bar{l}}_{(sq)}})=(p_{\bar{\bar{j}}_{(rq)}}p_{\bar{\bar{l}}_{(sq)}}\varepsilon_{\bar{i}_{(p)}}\varepsilon_{\bar{k}_{(p)}})=$$
$$={}^{0,0}\left(P_rP_s\varepsilon\varepsilon\right)=Z=(z_{\bar{\bar{j}}_{(rq)},\bar{\bar{l}}_{(sq)},\bar{i}_{(p)},\bar{k}_{(p)}}),$$

then $u_{\bar{i}_{(p)},\bar{\bar{j}}_{(rq)},\bar{k}_{(p)},\bar{\bar{l}}_{(sq)}}=z_{\bar{\bar{j}}_{(rq)},\bar{\bar{l}}_{(sq)},\bar{i}_{(p)},\bar{k}_{(p)}}$. This means, that the matrices $U$ and $Z$ are connected to each other by the transpose operation

$$U=Z^{T_1(r,s)}$$

with the transpose substitution $T_1(r,s)$:

$$T_1(r,s)=\begin{pmatrix}\bar{i}_{(p)},\bar{\bar{j}}_{(rq)},\bar{k}_{(p)},\bar{\bar{l}}_{(sq)}\\ \bar{\bar{j}}_{(rq)},\bar{\bar{l}}_{(sq)},\bar{i}_{(p)},\bar{k}_{(p)}\end{pmatrix}.$$

The expression (42) is represented now in the following form:

$$V_{\hat{\beta}_{r,s}}=(v_{\beta,\bar{i}_{(p)},\bar{\bar{j}}_{(rq)},\bar{k}_{(p)},\bar{\bar{l}}_{(sq)}})=\frac{1}{l^2}{}^{0,0}\left(\left(\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)P_s(x_i)\right)\right)V_\varepsilon\right)^{T_1(r,s)}. \tag{43}$$

Due to the orthogonality condition (15) of the polynomials $P_r(x_i)$ we have that $V_{\hat{\beta}_{r,s}}=0$ providing $r\neq s$. Thus, the estimations $\hat{\beta}_{(p,rq)}$ are uncorrelated. This is an important positive feature of the orthogonal regression compared to the mdm regression [18].

The variation $V_{\hat{\beta}_r}=\mathrm{cov}(\hat{\beta}_{(p,rq)},\hat{\beta}_{(p,rq)})$ of the estimation $\hat{\beta}_{(p,rq)}$ is defined as the matrix $V_{\hat{\beta}_{r,r}}$ by one of the following expressions:

$$V_{\hat{\beta}_r}=V_{\hat{\beta}_{r,r}}=\left(v^{(\hat{\beta}_r)}_{\bar{i}_{(p)},\bar{\bar{j}}_{(rq)},\bar{k}_{(p)},\bar{\bar{l}}_{(rq)}}\right)=\frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)V_\varepsilon P_r(x_i)\right)^{T(r,r)},$$

or

$$V_{\hat{\beta}_r}=V_{\hat{\beta}_{r,r}}=\left(v^{(\hat{\beta}_r)}_{\bar{i}_{(p)},\bar{\bar{j}}_{(rq)},\bar{k}_{(p)},\bar{\bar{l}}_{(rq)}}\right)=\frac{1}{l^2}{}^{0,0}\left(\left(\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)P_r(x_i)\right)\right)V_\varepsilon\right)^{T_1(r,r)}. \tag{44}$$

We can represent the data observations model (24) by the polynomials $P_r(x_i)$ (32) and receive the estimation $\hat{\alpha}_{(p,rq)}$ of the coefficient $\alpha_{(p,rq)}$ as follows:

$$\hat{\alpha}_{(p,rq)}=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(y_{o,i}Q_r(x_i)\right),\quad r=0,1,2,...,m. \tag{45}$$

The properties of the estimation $\widehat{\alpha}_{(p,rq)}$ (45) can be received analogously to the estimation $\widehat{\beta}_{(p,rq)}$:

$$\widehat{\alpha}_{(p,rq)} = \alpha_{(p,rq)} + \frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i Q_r(x_i)\right),$$

$$E_{\widehat{\alpha}(p,rq)} = E(\widehat{\alpha}_{(p,rq)}) = E\left(\alpha_{(p,rq)} + \frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i Q_r(x_i)\right)\right) = \alpha_{(p,rq)}, \tag{46}$$

$$\widehat{\alpha}_{(p,rq)} - E(\widehat{\alpha}_{(p,rq)}) = \widehat{\alpha}_{(p,rq)} - \alpha_{(p,rq)} = \frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i Q_r(x_i)\right) = \overset{\circ}{\widehat{\alpha}}_{(p,rq)},$$

$$V_{\widehat{\alpha}r,s} = \operatorname{cov}(\widehat{\alpha}_{(p,rq)}, \widehat{\alpha}_{(p,sq)}) = \frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)E(\varepsilon_i\varepsilon_i)Q_s(x_i)\right)^T = \frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)V_\varepsilon Q_s(x_i)\right)^{T(r,s)},$$

$$V_{\widehat{\alpha}r,s} = \operatorname{cov}(\widehat{\alpha}_{(p,rq)}, \widehat{\alpha}_{(p,sq)}) = \frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)Q_s(x_i)V_\varepsilon\right)^{T_1} = \frac{1}{l^2}{}^{0,0}\left(\left(\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)Q_s(x_i)\right)\right)V_\varepsilon\right)^{T_1(r,s)},$$

$$V_{\widehat{\alpha}r,s} = \operatorname{cov}(\widehat{\alpha}_{(p,rq)}, \widehat{\alpha}_{(p,sq)}) = 0 \ \text{ providing } \ r \neq s,$$

$$V_{\widehat{\alpha}r} = \left(v^{(\widehat{\alpha}r)}_{\overline{i}(p),\overline{\overline{j}}(rq),\overline{k}(p),\overline{\overline{l}}(rq)}\right) = \operatorname{cov}(\widehat{\alpha}_{(p,rq)}, \widehat{\alpha}_{(p,rq)}) = E\left(\left(\widehat{\alpha}_{(p,rq)} - E(\widehat{\alpha}_{(p,rq)})\right)^2\right) =$$

$$= \frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)V_\varepsilon Q_r(x_i)\right)^{T(r,r)},$$

$$V_{\widehat{\alpha}r} = \left(v^{(\widehat{\alpha}r)}_{\overline{i}(p),\overline{\overline{j}}(rq),\overline{k}(p),\overline{\overline{l}}(rq)}\right) = \operatorname{cov}(\widehat{\alpha}_{(p,rq)}, \widehat{\alpha}_{(p,rq)}) = E\left(\left(\widehat{\alpha}_{(p,rq)} - E(\widehat{\alpha}_{(p,rq)})\right)^2\right) =$$

$$= \frac{1}{l^2}{}^{0,0}\left(\left(\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)Q_r(x_i)\right)\right)V_\varepsilon\right)^{T_1(r,r)}. \tag{47}$$

Let us formulate the result as follows.

T h e o r e m 2. *Let the errors* $\varepsilon_i$ *in the measurements model* (24) *have zero mean value* $E(\varepsilon_i) = 0$, *the covariance matrix* $V_\varepsilon = E({}^{0,0}\varepsilon_i^2) = E({}^{0,0}\varepsilon^2)$ *and are independent by i. Then*

*1) the estimations* $\widehat{\beta}_{(p,rq)}$ *of the coefficients* $\beta_{(p,rq)}$, $r = 0,1,\ldots,m$, *in the presentation* (28) *are unbiased, i. e.* $E_{\widehat{\beta}(p,rq)} = E(\widehat{\beta}_{(p,rq)}) = \beta_{(p,rq)}$, *are uncorrelated by r, i. e.* $V_{\widehat{\beta}r,s} = \operatorname{cov}(\widehat{\beta}_{(p,rq)}, \widehat{\beta}_{(p,sq)}) = 0$ *provided r ≠ s, and have the variation* $V_{\widehat{\beta}r} = \operatorname{cov}(\widehat{\beta}_{(p,rq)}, \widehat{\beta}_{(p,rq)})$ (44);

*2) the estimations* $\widehat{\alpha}_{(p,rq)}$ *of the coefficients* $\alpha_{(p,rq)}$, $r = 0,1,\ldots,m$, *in the presentation* (35) *are unbiased, i. e.* $E_{\widehat{\alpha}(p,rq)} = E(\widehat{\alpha}_{(p,rq)}) = \alpha_{(p,rq)}$, *are uncorrelated by r, i. e.* $V_{\widehat{\alpha}r,s} = \operatorname{cov}(\widehat{\alpha}_{(p,rq)}, \widehat{\alpha}_{(p,sq)}) = 0$ *provided r ≠ s, and have the variation* $V_{\widehat{\alpha}r} = \operatorname{cov}(\widehat{\alpha}_{(p,rq)}, \widehat{\alpha}_{(p,rq)})$ (47).

***2.3. Mutual properties of the estimations of the regression coefficients.*** We get analogously (41), (42), (43)

$$\operatorname{cov}(\widehat{\alpha}_{(p,rq)}, \widehat{\beta}_{(p,sq)}) = \frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)V_\varepsilon P_s(x_i)\right)^{T(r,s)},$$

or

$$\operatorname{cov}(\widehat{\alpha}_{(p,rq)}, \widehat{\beta}_{(p,sq)}) = \frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)P_s(x_i)V_\varepsilon\right)^{T_1(r,s)}.$$

Since (16)

$$\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)P_k(x_i)\right) = \begin{cases} 0, & k \neq r, \\ r!E^{(s)}(0,rq), & k = r, \end{cases}$$

then

$$\operatorname{cov}(\widehat{\alpha}_{(p,rq)},\widehat{\beta}_{(p,sq)})=\begin{cases}0, & r\neq s,\\ \dfrac{r!}{l}\,{}^{0,0}\!\left(E^{(s)}(0,rq)V_\varepsilon\right)^{T_1(r,r)}, & r=s,\end{cases} \tag{48}$$

where $T_1(r,s)$ is the following substitution of transpose:

$$T_1(r,s)=\begin{pmatrix}\overline{i}_{(p)},\overline{j}_{(r)},\overline{k}_{(p)},\overline{l}_{(r)}\\ \overline{j}_{(r)},\overline{l}_{(r)},\overline{i}_{(p)},\overline{k}_{(p)}\end{pmatrix}. \tag{49}$$

Let us consider also the product ${}^{0,0}\!\left(\overset{\circ}{\widehat{\beta}}_{(rq,p)}\overset{\circ}{\widehat{\beta}}_{(p,sq)}\right)$. Since $\dfrac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i P_r(x_i)\right)=\overset{\circ}{\widehat{\beta}}_{(p,rq)}$, then

$${}^{0,0}\!\left(\overset{\circ}{\widehat{\beta}}_{(rq,p)}\overset{\circ}{\widehat{\beta}}_{(p,sq)}\right)={}^{0,0}\!\left(\left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)\varepsilon_i\right)\right)\left(\frac{1}{l}\sum_{j=1}^{l}{}^{0,0}\left(\varepsilon_j P_s(x_j)\right)\right)\right)=$$

$$=\frac{1}{l^2}\sum_{i=1}^{l}\sum_{j=1}^{l}{}^{0,0}\!\left({}^{0,0}\left(P_r(x_i)\varepsilon_i\right){}^{0,0}\left(\varepsilon_j P_s(x_j)\right)\right)=\frac{1}{l^2}\sum_{i=1}^{l}\sum_{j=1}^{l}{}^{0,0}\!\left(P_r(x_i)\,{}^{0,0}\left({}^{0,0}(\varepsilon_i\varepsilon_j)P_s(x_j)\right)\right)$$

and

$$V'_{\widehat{\beta}_{r,s}}=E\left({}^{0,0}(\overset{\circ}{\widehat{\beta}}_{(rq,p)}\overset{\circ}{\widehat{\beta}}_{(p,sq)})\right)=\frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\!\left(P_r(x_i)\,{}^{0,0}\left(V_\varepsilon P_s(x_i)\right)\right)=\operatorname{cov}(\widehat{\beta}_{(rq,p)},\widehat{\beta}_{(p,sq)}).$$

Due to the condition (15) we have

$$V'_{\widehat{\beta}_{r,s}}=\operatorname{cov}(\widehat{\beta}_{(rq,p)},\widehat{\beta}_{(p,sq)})=\begin{cases}0, & r\neq s,\\ \dfrac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\!\left(P_r(x_i)\,{}^{0,0}\left(V_\varepsilon P_r(x_i)\right)\right), & r=s.\end{cases} \tag{50}$$

The comparison of the expressions (50) and (42) shows that $(V_{\widehat{\beta}_{r,s}})^{T^{-1}(r,s)}=V'_{\widehat{\beta}_{r,s}}$, where $T^{-1}(r,s)$ is the substitution inverse to the substitution $T(r,s)$ (40). Since $T(r,s)=(B_{p+rq,p},E_{p+sq})$, then $T^{-1}(r,s)=(B_{p+rq,rq},E_{p+sq})$. We will denote $T^{-1}(r,s)=T^*(r,s)$. Thus, we have the following expression:

$$V'_{\widehat{\beta}_{r,s}}=\operatorname{cov}(\widehat{\beta}_{(rq,p)},\widehat{\beta}_{(p,sq)})=\begin{cases}0, & r\neq s,\\ (V_{\widehat{\beta}_r})^{T^*(r,r)}, & r=s,\end{cases} \tag{51}$$

where $T^*(r,r)=(B_{p+rq,rq},E_{p+rq})$.

T h e o r e m 3. *Under the conditions of Theorem 2*

*1) the covariance* $\operatorname{cov}(\widehat{\alpha}_{(p,rq)},\widehat{\beta}_{(p,sq)})$ *between the estimations* $\widehat{\alpha}_{(p,rq)}$ *and* $\widehat{\beta}_{(p,sq)}$ *is defined by the expressions* (48), (49);

*2) the covariance* $\operatorname{cov}(\widehat{\beta}_{(rq,p)},\widehat{\beta}_{(p,sq)})$ *between the estimations* $\widehat{\beta}_{(rq,p)}$ *and* $\widehat{\beta}_{(p,sq)}$ *is defined by the expressions* (51).

***2.4. Properties of the estimation of the response.*** The estimation $\widehat{y}$ of the response $y$ in (26) is defined as follows (see (28), (35)):

$$\widehat{y}\sim\sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\!\left(\widehat{\beta}_{(p,rq)}Q_r(x)\right), \tag{52}$$

$$\widehat{y}\sim\sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\!\left(\widehat{\alpha}_{(p,rq)}P_r(x)\right). \tag{53}$$

Since $E(\widehat{\beta}_{(p,rq)})=\beta_{(p,rq)}$, $E(\widehat{\alpha}_{(p,rq)})=\alpha_{(p,rq)}$ then

$$E(\widehat{y})=E\left(\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\widehat{\beta}_{(p,rq)}Q_r(x)\right)\right)=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(E(\widehat{\beta}_{(p,rq)})Q_r(x)\right)=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\beta_{(p,rq)}Q_r(x)\right)=y,$$

$$E(\widehat{y})=E\left(\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\widehat{\alpha}_{(p,rq)}H_r(x)\right)\right)=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(E(\widehat{\alpha}_{(p,rq)})H_r(x)\right)=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\alpha_{(p,rq)}H_r(x)\right)=y.$$

Thus, the estimation $\widehat{y}$ (52), (53) of the response $y$ is unbiased. Further,

$$\overset{\circ}{\widehat{y}}=\widehat{y}-E(\widehat{y}))=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(Q_r(x)\overset{\circ}{\widehat{\beta}}_{(rq,p)}\right)=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\overset{\circ}{\widehat{\beta}}_{(p,rq)}Q_r(x)\right),$$

and

$$\left(\widehat{y}-E(\widehat{y})\right)^2={}^{0,0}\left(\left(\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(Q_r(x)\overset{\circ}{\widehat{\beta}}_{(rq,p)}\right)\right)\left(\sum_{s=0}^{m}\frac{1}{s!}{}^{0,sq}\left(\overset{\circ}{\widehat{\beta}}_{(p,sq)}Q_s(x)\right)\right)\right)=$$

$$=\sum_{r=0}^{m}\frac{1}{r!}\sum_{s=0}^{m}\frac{1}{s!}{}^{0,0}\left(\left({}^{0,rq}\left(Q_r(x)\overset{\circ}{\widehat{\beta}}_{(rq,p)}\right)\right)\left({}^{0,sq}\left(\overset{\circ}{\widehat{\beta}}_{(p,sq)}Q_s(x)\right)\right)\right)=$$

$$=\sum_{r=0}^{m}\frac{1}{r!}\sum_{s=0}^{m}\frac{1}{s!}{}^{0,rq}\left(Q_r(x){}^{0,sq}\left({}^{0,0}\left(\overset{\circ}{\widehat{\beta}}_{(rq,p)}\overset{\circ}{\widehat{\beta}}_{(p,sq)}\right)Q_s(x)\right)\right).$$

Due to the expression (50), the covariance matrix $V_{\widehat{y}}$ of the estimation $\widehat{y}$ of the output variable $y$ is defined by the following expression:

$$V_{\widehat{y}}=(v^{(y)}_{\overline{i}(p),\overline{j}(p)})=E\left(\left(\widehat{y}-E(\widehat{y})\right)^2\right)=\sum_{r=0}^{m}\frac{1}{(r!)^2}{}^{0,rq}\left(Q_r(x){}^{0,rq}\left(V'_{\widehat{\beta}_{r,r}}Q_r(x)\right)\right),\tag{54}$$

where $V'_{\widehat{\beta}_{r,r}}$ is defined by the expression (51), i. e.

$$V'_{\widehat{\beta}_{r,r}}=\mathrm{cov}(\widehat{\beta}_{(rq,p)},\widehat{\beta}_{(p,rq)})=(V_{\widehat{\beta}_{r,r}})^{T^*(r,r)},\quad T^*(r,r)=(B_{p+rq,rq},E_{p+rq}).$$

The expression (54) shows that the covariance matrix $V_{\widehat{y}}$ of the estimation $\widehat{y}$ of the output variable $y$ is defined by each term of Fourier series separately.

T h e o r e m 4. *Under the conditions of Theorem 2*

*1) the estimation $\widehat{y}$ of the respond y is unbiased, i. e. $E(\widehat{y})=y$;*

*2) the covariance matrix $V_{\widehat{y}}=E\left(\left(\widehat{y}-E(\widehat{y})\right)^2\right)$ of the estimation $\widehat{y}$ is defined by the expression* (54).

***2.5. Properties of the estimation*** $\widehat{V}_\varepsilon$. Let us find the mathematical expectation $E(\widehat{V}_\varepsilon)$ of the estimation $\widehat{V}_\varepsilon$ (36). Since $y_{o,i}=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\beta_{(p,rq)}Q_r(x_i)\right)+\varepsilon_i$, then $E(y_{o,i})=\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\beta_{(p,rq)}Q_r(x_i)\right)$, $y_{o,i}-E(y_{o,i})=\varepsilon_i$ and

$$\widehat{V}_\varepsilon=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(y_{o,i}-\widehat{y}_i\right)^2=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\left(y_{o,i}-E(y_{o,i})\right)-\left(\widehat{y}_i-E(y_{o,i})\right)\right)^2=$$

$$=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i-\left(\widehat{y}_i-E(y_{o,i})\right)\right)^2=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i-\sum_{r=0}^{m}\frac{1}{r!}{}^{0,rq}\left(\overset{\circ}{\widehat{\beta}}_{(p,rq)}Q_r(x_i)\right)\right)^2,\tag{55}$$

where $\overset{\circ}{\hat{\beta}}_{(p,rq)} = \hat{\beta}_{(p,rq)} - E(\hat{\beta}_{(p,rq)})$ is defined by the expression (46). Further, we can write the formula (55) in other form:

$$\hat{V}_\varepsilon = \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\varepsilon_i - \sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\overset{\circ}{\hat{\beta}}_{(p,rq)}\, Q_r(x_i)\right)\right)^2 =$$

$$= \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\left(\varepsilon_i - \sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\overset{\circ}{\hat{\alpha}}_{(p,rq)}\, P_r(x_i)\right)\right)\left(\varepsilon_i - \sum_{s=0}^{m}\frac{1}{s!}\,{}^{0,sq}\left(Q_s(x_i)\overset{\circ}{\hat{\beta}}_{(sq,p)}\right)\right)\right), \tag{56}$$

where $\overset{\circ}{\hat{\beta}}_{(sq,p)} = (\overset{\circ}{\hat{\beta}}_{(p,sq)})^{B_{p+sq,sq}}$. Transformation of the expression (56) gives

$$\hat{V}_\varepsilon = \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}(\varepsilon_i\varepsilon_i) - \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\varepsilon_i \sum_{s=0}^{m}\frac{1}{s!}\,{}^{0,sq}\left(Q_s(x_i)\overset{\circ}{\hat{\beta}}_{(sq,p)}\right)\right) - \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\left(\sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\overset{\circ}{\hat{\alpha}}_{(p,rq)}\, P_r(x_i)\right)\right)\varepsilon_i\right) +$$

$$+ \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\left(\sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\overset{\circ}{\hat{\alpha}}_{(p,rq)}\, P_r(x_i)\right)\right)\left(\sum_{s=0}^{m}\frac{1}{s!}\,{}^{0,sq}\left(Q_s(x_i)\overset{\circ}{\hat{\beta}}_{(sq,p)}\right)\right)\right) = s_1 - s_2 - s_3 + s_4. \tag{57}$$

We have for the first summand $s_1$ of the expression (57) that

$$E(s_1) = E\left(\frac{1}{l}\sum_{i=1}^{l} {}^{0,0}(\varepsilon_i\varepsilon_i)\right) = V_\varepsilon.$$

Let us consider the summand $s_2$ of the expression (57)

$$s_2 = \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\varepsilon_i \sum_{s=0}^{m}\frac{1}{s!}\,{}^{0,sq}\left(Q_s(x_i)\overset{\circ}{\hat{\beta}}_{(sq,p)}\right)\right) = \frac{1}{l}\sum_{i=1}^{l} {}^{0,0}\left(\varepsilon_i \sum_{s=0}^{m}\frac{1}{s!}\,{}^{0,sq}\left(Q_s(x_i)\left(\frac{1}{l}\sum_{j=1}^{l} {}^{0,0}\left(P_s(x_j)\varepsilon_j\right)\right)\right)\right) =$$

$$= \sum_{s=0}^{m}\frac{1}{s!}\frac{1}{l^2}\sum_{i=1}^{l}\sum_{j=1}^{l} {}^{0,0}\left(\varepsilon_i\ {}^{0,0}\left({}^{0,sq}\left(Q_s(x_i)P_s(x_j)\right)\varepsilon_j\right)\right). \tag{58}$$

The expression ${}^{0,0}\left(\varepsilon_i\ {}^{0,0}\left({}^{0,sq}\left(Q_s(x_i)P_s(x_j)\right)\varepsilon_j\right)\right)$ in (58) can be transformed. Let us denote ${}^{0,sq}\left(Q_s(x_i)P_s(x_j)\right) = m_{i,j}$. Since $m_{i,j}$ is a scalar, then

$${}^{0,0}\left(\varepsilon_i\ {}^{0,0}\left({}^{0,sq}\left(Q_s(x_i)P_s(x_j)\right)\varepsilon_j\right)\right) = {}^{0,0}\left(\varepsilon_i\ {}^{0,0}(m_{i,j}\varepsilon_j)\right) = {}^{0,0}\left(\varepsilon_i\ {}^{0,0}(\varepsilon_j m_{i,j})\right) = {}^{0,0}\left({}^{0,0}(\varepsilon_i\,\varepsilon_j)m_{i,j}\right) \tag{59}$$

and

$$s_2 = \sum_{s=0}^{m}\frac{1}{s!}\frac{1}{l^2}\sum_{i=1}^{l}\sum_{j=1}^{l} {}^{0,0}\left(\varepsilon_i\ {}^{0,0}(m_{i,j}\varepsilon_j)\right) = \sum_{s=0}^{m}\frac{1}{s!}\frac{1}{l^2}\sum_{i=1}^{l}\sum_{j=1}^{l} {}^{0,0}\left({}^{0,0}(\varepsilon_i\varepsilon_j)m_{i,j}\right). \tag{60}$$

The mathematical expectation of the expression (59) is equal:

$$E\left({}^{0,0}\left({}^{0,0}(\varepsilon_i\,\varepsilon_j)m_{i,j}\right)\right) = {}^{0,0}\left(E\left({}^{0,0}(\varepsilon_i\,\varepsilon_j)\right)m_{i,j}\right).$$

Due to the independency of $\varepsilon_i$ and $\varepsilon_j$ when $i \neq j$ we have

$$E\left({}^{0,0}\left({}^{0,0}(\varepsilon_i\,\varepsilon_j)m_{i,j}\right)\right) = {}^{0,0}\left(E\left({}^{0,0}(\varepsilon_i\,\varepsilon_j)\right)m_{i,j}\right) = \begin{cases} {}^{0,0}(V_\varepsilon m_{i,i}), & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$E(s_2)=\sum_{s=0}^{m}\frac{1}{r!}\frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(V_\varepsilon m_{i,i}\right)=\sum_{s=0}^{m}\frac{1}{s!}\frac{1}{l^2}\sum_{i=1}^{l}{}^{0,0}\left(V_\varepsilon\,{}^{0,sq}\left(Q_s(x_i)P_s(x_i)\right)\right).$$

Due to the property (A2), which was proved in theorem A1 in Appendix, we have

$$E(s_2)=\frac{V_\varepsilon}{l}\sum_{s=0}^{m}\operatorname{tr}\left(E^{(s)}(0,sq)\right).$$

Let us consider the summand $s_3$ of the expression (57). Since

$$\mathring{\hat{\alpha}}_{(p,rq)}=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i Q_r(x_i)\right)$$

then

$$s_3=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\left(\sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\mathring{\hat{\alpha}}_{(p,rq)}P_r(x_i)\right)\right)\varepsilon_i\right)=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\left(\sum_{r=0}^{m}\frac{1}{r!}\frac{1}{l}\sum_{j=1}^{l}{}^{0,rq}\left({}^{0,0}\left(\varepsilon_j Q_r(x_j)\right)P_r(x_i)\right)\right)\varepsilon_i\right)=$$

$$=\sum_{r=0}^{m}\frac{1}{r!}\frac{1}{l^2}\sum_{i=1}^{l}\sum_{j=1}^{l}{}^{0,0}\left(\varepsilon_j\,{}^{0,0}\left({}^{0,rq}\left(Q_r(x_j)P_r(x_i)\right)\varepsilon_i\right)\right)=s_2.$$

We have

$$E(s_3)=E(s_2)=\frac{V_\varepsilon}{l}\sum_{r=0}^{m}\operatorname{tr}\left(E^{(s)}(0,rq)\right).$$

Let us consider the summand $s_4$ of the expression (57):

$$s_4=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\left(\sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\mathring{\hat{\alpha}}_{(p,rq)}P_r(x_i)\right)\right)\left(\sum_{s=0}^{m}\frac{1}{s!}\,{}^{0,sq}\left(Q_s(x_i)\mathring{\hat{\beta}}_{(sq,p)}\right)\right)\right)=$$

$$=\frac{1}{l}\sum_{i=1}^{l}\left(\sum_{r=0}^{m}\sum_{s=0}^{m}\frac{1}{r!}\frac{1}{s!}\,{}^{0,rq}\left(\mathring{\hat{\alpha}}_{(p,rq)}\,{}^{0,sq}\left({}^{0,0}\left(P_r(x_i)Q_s(x_i)\right)\mathring{\hat{\beta}}_{(sq,p)}\right)\right)\right).$$

Due to the property (16) of the polynomials $Q_r(x)$ and $P_r(x)$ we have

$$s_4=\sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\mathring{\hat{\alpha}}_{(p,rq)}\,{}^{0,sq}\left(E^{(s)}(0,rq)\mathring{\hat{\beta}}_{(rq,p)}\right)\right)=\sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\mathring{\hat{\alpha}}_{(p,rq)}\mathring{\hat{\beta}}_{(rq,p)}\right).$$

Since $\mathring{\hat{\alpha}}_{(p,rq)}=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i Q_r(x_i)\right)$, $\mathring{\hat{\beta}}_{(rq,p)}=\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)\varepsilon_i\right)$, then

$$s_4=\sum_{r=0}^{m}\frac{1}{r!}\,{}^{0,rq}\left(\left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i Q_r(x_i)\right)\right)\left(\frac{1}{l}\sum_{j=1}^{l}{}^{0,0}\left(P_r(x_j)\varepsilon_j\right)\right)\right)=$$

$$=\sum_{r=0}^{m}\frac{1}{r!}\frac{1}{l^2}\sum_{i=1}^{l}\sum_{j=1}^{l}{}^{0,0}\left(\varepsilon_i\,{}^{0,0}\left({}^{0,rq}\left(Q_r(x_i)P_r(x_j)\right)\varepsilon_j\right)\right)=s_2.$$

We get

$$E(s_4)=E(s_2)=\frac{V_\varepsilon}{l}\sum_{r=0}^{m}\operatorname{tr}\left(E^{(s)}(0,rq)\right).$$

Since $s_2 = s_3 = s_4$, we have as a result

$$E(\widehat{V}_\varepsilon) = E(s_1 - s_2 - s_3 + s_4) = E(s_1 - s_2) = V_\varepsilon - \frac{V_\varepsilon}{l}\sum_{r=0}^{m} \operatorname{tr}\left(E^{(s)}(0, rq)\right).$$

Taking into account the formula (A6) (Appendix, theorem A2), we get

$$E(\widehat{V}_\varepsilon) = V_\varepsilon - \frac{V_\varepsilon}{l}\sum_{r=0}^{m} \frac{(n+r-1)!}{r!(n-1)!}, \tag{61}$$

where $n = n_1 n_2 \cdots n_q$ is the length of the $q$-dimensional-matrix argument $x$. Thus, the estimation $\widehat{V}_\varepsilon$ is biased. If we denote $d_s = \sum_{r=0}^{m} \frac{(n+r-1)!}{r!(n-1)!}$, then $E(\widehat{V}_\varepsilon) = V_\varepsilon - \frac{d_s}{l} V_\varepsilon$, where $\frac{-d_s}{l} V_\varepsilon$ is bias of the estimation $\widehat{V}_\varepsilon$. The unbiased estimation of $\widehat{V}_\varepsilon$ is

$$\widehat{V}_{\varepsilon,1} = \widehat{V}_\varepsilon \Big/ \left(-\frac{d_s}{l} + 1\right) = \frac{\widehat{V}_\varepsilon l}{l - d_s}. \tag{62}$$

The number *ds* is the missing number of the freedom degreases.

Instance. Regression function is a 3 degree polynomial of the vector argument with 3 components: $m = 3$, $n = 3$. In this case

$$d_s = \sum_{r=0}^{3} \frac{(n+r-1)!}{r!(n-1)!} = \sum_{r=0}^{3} \frac{(r+2)!}{r!2!} = \frac{2!}{0!2!} + \frac{3!}{1!2!} + \frac{4!}{2!2!} + \frac{5!}{3!2!} = 1 + 3 + 6 + 10 = 20.$$

Obviously, the estimation $\widehat{V}_\varepsilon$ can be calculated by the formula (36) in which the estimation $\widehat{y}_i$ is calculated by the formula

$$\widehat{y}_i = (\widehat{y}_{\bar{i}(p)}) = \sum_{r=0}^{m} \frac{1}{r!}\, {}^{0,rq}\left(\widehat{\alpha}_{(p,rq)} P_r(x_i)\right), \quad i = 1, 2, \ldots, l, \quad \bar{i}_{(p)} = i_1, i_2, \ldots, i_p,$$

instead of the formula (28).

Theorem 5. *Under the conditions of Theorem 2*

*1) the mean value* $E(\widehat{V}_\varepsilon)$ *of the estimation* $\widehat{V}_\varepsilon$ *of the covariance matrix* $V_\varepsilon = E({}^{0,0}\varepsilon^2)$ *of the errors* $\varepsilon$ *is defined by the expression* (61) *and is biased;*

*2) the unbiased estimation of* $\widehat{V}_\varepsilon$ *is defined by the expression* (62).

***2.6. Properties of the estimations of coefficients in multidimensional-matrix representation.*** The estimations $\widehat{c}_{(p,kq)}$ of the coefficients $c_{(p,kq)}$ in representation of the regression function $y(x)$ by the degrees of the argument (17), (27) are defined by the expressions (31) or (34). Let us use the following expression (31):

$$\widehat{c}_{(p,rq)} = (\widehat{c}_{\bar{i}(p), \bar{\bar{j}}(rq)}) = \sum_{k=r}^{m} \frac{1}{k!}\, {}^{0,kq}\left(\widehat{\beta}_{(p,kq)} C^*_{(k,rq)}\right), \quad r = 0, 1, 2, .., m,$$

and obtain the mathematical expectation $E(\widehat{c}_{(p,rq)})$ of the estimation $\widehat{c}_{(p,rq)}$ and the covariance matrix $\operatorname{cov}(\widehat{c}_{(p,rq)}, \widehat{c}_{(p,sq)})$ of the estimations $\widehat{c}_{(p,rq)}$, $\widehat{c}_{(p,sq)}$. Since

$$E\left({}^{0,kq}(\widehat{\beta}_{(p,kq)} C^*_{(k,rq)})\right) = {}^{0,kq}\left(E(\widehat{\beta}_{(p,kq)}) C^*_{(k,rq)}\right) = {}^{0,kq}(\beta_{(p,kq)} C^*_{(k,rq)}),$$

then

$$E(\widehat{c}_{(p,rq)}) = \sum_{k=r}^{m} \frac{1}{k!}\, {}^{0,kq}\left(\beta_{(p,kq)} C^*_{(k,rq)}\right) = c_{(p,rq)}, \quad r = 0, 1, 2, .., m.$$

It means that $\widehat{c}_{(p,rq)}$ is unbiased estimation of the coefficient $c_{(p,rq)}$. We will denote $\mathring{c}_{(p,rq)} = \widehat{c}_{(p,rq)} - c_{(p,rq)}$. Then $\mathring{c}_{(p,rq)} = \sum_{k=r}^{m} \frac{1}{k!}\, {}^{0,kq}\left(\mathring{\widehat{\beta}}_{(p,kq)} C^*_{(k,rq)}\right)$. Further,

$$V_{\hat{c}_{r,s}} = \operatorname{cov}(\hat{c}_{(p,rq)}, \hat{c}_{(p,sq)}) = E\left( \overset{\circ}{\hat{c}}_{(p,rq)} \overset{\circ}{\hat{c}}_{(p,sq)} \right) =$$

$$= E\left( \sum_{k=r}^{m} \sum_{n=s}^{m} \frac{1}{k!n!} {}^{0,0}\left( {}^{0,kq}\left( \hat{\beta}_{(p,kq)} C^{*}_{(k,rq)} \right) {}^{0,nq}\left( \hat{\beta}_{(p,nq)} C^{*}_{(n,sq)} \right) \right) \right) =$$

$$= \left( v^{(\hat{c}_{r,s})}_{\bar{i}(p),\bar{\bar{j}}(rq),\bar{\lambda}(p),\bar{\bar{\mu}}(sq)} \right) = \sum_{k=r}^{m} \sum_{n=s}^{m} \frac{1}{k!n!} E\left( {}^{0,0}\left( {}^{0,kq}\left( \hat{\beta}_{(p,kq)} C^{*}_{(k,rq)} \right) {}^{0,nq}\left( \hat{\beta}_{(p,nq)} C^{*}_{(n,sq)} \right) \right) \right) =$$

$$= \sum_{k=r}^{m} \sum_{n=s}^{m} \frac{1}{k!n!} U_{k,n}. \tag{63}$$

Let us consider the separate summand $U_{k,n}$ in (63)

$$U_{k,n} = (u_{\bar{i}(p),\bar{\bar{j}}(rq),\bar{\lambda}(p),\bar{\bar{\mu}}(sq)}) = E\left( {}^{0,0}\left( {}^{0,kq}\left( \hat{\beta}_{(p,kq)} C^{*}_{(k,rq)} \right) {}^{0,nq}\left( \hat{\beta}_{(p,nq)} C^{*}_{(n,sq)} \right) \right) \right)$$

and transform it. First, we transform the multiplier ${}^{0,kq}\left( \hat{\beta}_{(p,kq)} C^{*}_{(k,rq)} \right)$:

$${}^{0,kq}\left( \hat{\beta}_{(p,kq)} C^{*}_{(k,rq)} \right) = (f_{\bar{i}(p),\bar{\bar{j}}(rq)}) = \left( \sum_{\bar{\bar{j}}(kq)} \beta_{\bar{i}(p),\bar{\bar{j}}(kq)} c^{*}_{\bar{\bar{j}}(kq),\bar{\bar{j}}(rq)} \right) = \left( \sum_{\bar{\bar{j}}(kq)} c^{*}_{\bar{\bar{j}}(kq),\bar{\bar{j}}(rq)} \beta_{\bar{i}(p),\bar{\bar{j}}(kq)} \right) =$$

$$= \left( \sum_{\bar{\bar{j}}(kq)} c'_{\bar{\bar{j}}(rq),\bar{\bar{j}}(kq)} \ \beta'_{\bar{\bar{j}}(kq),\bar{i}(p)} \right) = (f'_{\bar{\bar{j}}(rq),\bar{i}(p)}) = {}^{0,kq}\left( C'_{p,kq} \hat{\beta}'_{(p,kq)} \right).$$

Since $c'_{\bar{\bar{j}}(rq),\bar{\bar{j}}(kq)} = c^{*}_{\bar{\bar{j}}(kq),\bar{\bar{j}}(rq)}$, then $C'_{(k,rq)} = \left( C^{*}_{(k,rq)} \right)^{B_{k+rq,rq}}$. Since $\beta'_{\bar{\bar{j}}(kq),\bar{i}(p)} = \beta_{\bar{i}(p),\bar{\bar{j}}(kq)}$, then $\hat{\beta}'_{(p,kq)} = \left( \hat{\beta}_{(p,kq)} \right)^{B_{p+kq,kq}}$. Since $f'_{\bar{\bar{j}}(rq),\bar{i}(p)} = f_{\bar{i}(p),\bar{\bar{j}}(rq)}$, then $F = (F')^{B_{p+rq,p}}$. Thus, we have

$${}^{0,kq}\left( \hat{\beta}_{(p,kq)} C^{*}_{(k,rq)} \right) = {}^{0,kq}\left( \left( C^{*}_{(k,rq)} \right)^{B_{k+rq,rq}} \left( \hat{\beta}_{(p,kq)} \right)^{B_{p+kq,kq}} \right)^{B_{p+rq,p}}.$$

However,

$$E\left( {}^{0,0}\left( (\hat{\beta}_{(p,kq)})^{B_{p+kq,kq}} \hat{\beta}_{(p,nq)} \right) \right) = E\left( {}^{0,0} (\hat{\beta}_{(p,kq)} \hat{\beta}_{(p,nq)})^{(B_{p+kq,kq}, E_{p+nq})} \right) = (V_{\hat{\beta}_{k,n}})^{(B_{p+kq,kq}, E_{p+nq})}.$$

Then

$$U_{k,n} = {}^{0,kq}\left( (C^{*}_{(k,rq)})^{B_{k+rq,rq}} \, {}^{0,nq}\left( (V_{\hat{\beta}_{k,n}})^{(B_{p+kq,kq}, E_{p+nq})} C^{*}_{(n,sq)} \right) \right)^{(B_{p+rq,p}, E_{p+sq})}.$$

Let us note:

$$T_3(k,r) = B_{k+rq,rq},$$
$$T_2(r,s) = (B_{p+rq,p}, E_{p+sq}),$$
$$T_4(k,n) = (B_{p+kq,kq}, E_{p+nq}).$$

Then

$$U_{k,n} = {}^{0,kq}\left( (C^{*}_{(k,rq)})^{T_3(k,r)} \, {}^{0,nq}\left( (V_{\hat{\beta}_{k,n}})^{T_4(k,n)} C^{*}_{(n,sq)} \right) \right)^{T_2(r,s)}$$

and

$$V_{\hat{c}_{r,s}} = \left( v^{(\hat{c}_{l,s})}_{\bar{i}(p),\bar{\bar{j}}(rq),\bar{\lambda}(p),\bar{\bar{\mu}}(sq)} \right) = \sum_{k=r}^{m} \sum_{n=s}^{m} \frac{1}{k!n!} U_{k,n} =$$

$$= \sum_{k=r}^{m} \sum_{n=s}^{m} \frac{1}{k!n!} {}^{0,kq}\left( (C^{*}_{(k,rq)})^{T_3(k,r)} \, {}^{0,nq}\left( (V_{\hat{\beta}_{k,n}})^{T_4(k,n)} C^{*}_{(n,sq)} \right) \right)^{T_2(r,s)}.$$

Providing $V_{\widehat{\beta}_{k,n}} = \mathrm{cov}(\widehat{\beta}_{(p,kq)}, \widehat{\beta}_{(p,nq)}) = 0$ when $k \neq n$, we have

$$V_{\widehat{c}_{r,s}} = \left( v^{(\widehat{c}_{r,s})}_{\overline{i}(p),\overline{\overline{j}}(rq),\overline{\lambda}(p),\overline{\overline{\mu}}(sq)} \right) = \sum_{k=\max(r,s)}^{m} \frac{1}{(k!)^2} {}^{0,kq}\left( (C^*_{(k,rq)})^{T_3(k,r)} \, {}^{0,kq}\left( (V_{\widehat{\beta}_{k,k}})^{T_4(k,k)} C^*_{(k,sq)} \right) \right)^{T_2(r,s)}. \quad (64)$$

The covariance matrix of the estimation $\widehat{c}_{(p,kq)}$ is defined as $V_{\widehat{c}_{r,r}}$ :

$$V_{\widehat{c}_r} = V_{\widehat{c}_{r,r}} = \left( v^{(\widehat{c}_{r,r})}_{\overline{i}(p),\overline{\overline{j}}(r),\overline{\lambda}(p),\overline{\overline{\mu}}(r)} \right) = \mathrm{cov}(\widehat{c}_{(p,rq)}, \widehat{c}_{(p,rq)}) = E\left( \overset{\circ}{\widehat{c}}_{(p,rq)} \overset{\circ}{\widehat{c}}_{(p,rq)} \right) =$$

$$= \sum_{k=r}^{m} \frac{1}{(k!)^2} {}^{0,kq}\left( (C^*_{(k,rq)})^{T_3(k,r)} \, {}^{0,kq}\left( (V_{\widehat{\beta}_{k,k}})^{T_4(k,k)} C^*_{(k,rq)} \right) \right)^{T_2(r,r)}. \quad (65)$$

If we will use the expression (34)

$$\widehat{c}_{(p,rq)} = \sum_{k=r}^{m} \frac{1}{k!} {}^{0,kq}(\widehat{\alpha}_{(p,kq)} C_{(k,rq)}), \quad r = 0,1,2,..,m,$$

then we will have the following expression for $V_{\widehat{c}_{r,r}}$ :

$$V_{\widehat{c}_r} = V_{\widehat{c}_{r,r}} = \left( v^{(\widehat{c}_{r,r})}_{\overline{i}(p),\overline{\overline{j}}(rq),\overline{\lambda}(p),\overline{\overline{\mu}}(rq)} \right) = \sum_{k=r}^{m} \frac{1}{(k!)^2} {}^{0,kq}\left( (C_{(k,rq)})^{T_3(k,r)} \, {}^{0,kq}\left( (V_{\widehat{\alpha}_{k,k}})^{T_4(k,k)} C_{(k,rq)} \right) \right)^{T_2(r,r)}, \quad (66)$$

where $T_3(k,r) = B_{k+rq,rq}$, $T_2(r,r) = (B_{p+rq,p}, E_{p+rq})$, $T_4(k,k) = (B_{p+kq,kq}, E_{p+kq})$.

Let us remind that $C^*_{(k,rq)}$ is the coefficients of the polynomial $Q_k(x)$ of the degree $k$,

$$Q_k(x) = \sum_{r=0}^{k} {}^{0,rq}\left( C^*_{(k,rq)} x^r \right) = \sum_{r=0}^{k} {}^{0,rq}\left( x^r C^*_{(rq,k)} \right),$$

and $C_{(k,rq)}$ is the coefficient of the polynomial $P_k(x)$ of the degree $k$,

$$P_k(x) = \sum_{r=0}^{k} {}^{0,rq}\left( C_{(k,rq)} x^r \right) = \sum_{r=0}^{k} {}^{0,rq}\left( x^r C_{(rq,k)} \right).$$

If $rq = 0$, then $T_3(k,r) = B_{kq,0} = E_{kq}$, $T_2(r,r) = (B_{p,p}, E_p) = (E_p, E_p) = E_{2p}$. If $kq = 0$, then $T_4(k,k) = (B_{p,0}, E_p) = (E_p, E_p) = E_{2p}$. It means, that in these cases the transpose is not required for any $p = 0,1,2,...$ .

T h e o r e m 6. *Under the conditions of Theorem 2*

*1) the estimation* $\widehat{c}_{(p,rq)}$ *of the coefficient* $c_{(p,rq)}$ *in the representation* (17), (27) *of the regression function y*(*x*) *by the degrees of the argument is unbiased, i.e.* $E(\widehat{c}_{(p,rq)}) = c_{(p,rq)}$;

*2) the covariance* $V_{\widehat{c}_{r,s}} = \mathrm{cov}\left(\widehat{c}_{(p,rq)}, \widehat{c}_{(p,sq)}\right)$ *between the estimations* $\widehat{c}_{(p,rq)}$ *and* $\widehat{c}_{(p,sq)}$ *is defined by the expression* (64);

*3) the covariance matrix* $V_{\widehat{c}_r} = V_{\widehat{c}_{r,r}} = (v^{(\widehat{c}_{r,r})}_{\overline{i}(p),\overline{\overline{j}}(r),\overline{\lambda}(p),\overline{\overline{\mu}}(r)}) = \mathrm{cov}\left(\widehat{c}_{(p,rq)}, \widehat{c}_{(p,rq)}\right)$ *of the estimation* $\widehat{c}_{(p,rq)}$ *is defined by the expressions* (65) *or* (66).

***2.7. Distributions of the estimations.*** We will use the following notations for distributions: *N*(*A*,*R*) for the normal distribution with mathematical expectation *A* and covariance matrix *R*; *W*(*k*,*R*) for Wishart distribution with *k* degreases of freedom and parametric matrix *R*; *H*(*k*) for $\chi^2$ distribution with *k* degreases of freedom; *T*(*k*) for Student distribution with *k* degreases of freedom. We will denote the belonging of a random matrix ξ to some distribution by the relation symbol ∈. For instance, the notation $\xi \in H(k)$ means that the random matrix ξ has $\chi^2$ distribution with *k* degreases of freedom.

We will suppose that the errors $\varepsilon_i$ in the measurements model (24) have zero mean value $E(\varepsilon_i) = 0$, the covariance matrix $V_\varepsilon = E({}^{0,0}\varepsilon_i^2) = E({}^{0,0}\varepsilon^2)$, the normal distribution $N(0,V_\varepsilon)$, and are independent by *i*.

Let us start with the estimations $\widehat{\beta}_{(p,rq)}$ (30) and $V_{\widehat{\beta}_r}$ (44) and rewrite these expressions here:

$$\widehat{\beta}_{(p,rq)} = (\widehat{\beta}_{\overline{i}(p),\overline{\overline{j}}(rq)}) = \frac{1}{l} \sum_{i=1}^{l} {}^{0,0}\left( y_{o,i} P_r(x_i) \right), \quad r = 0,1,2,...,m, \quad (67)$$

$$V_{\widehat{\beta}_r} = \left( v^{(\widehat{\beta}_r)}_{\overline{i}_{(p)},\overline{\overline{j}}_{(rq)},\overline{k}_{(p)},\overline{\overline{l}}_{(rq)}} \right) = \frac{1}{l^2} {}^{0,0}\left( \left( \sum_{i=1}^{l} {}^{0,0}\left( P_r(x_i) P_r(x_i) \right) \right) V_\varepsilon \right)^{T_1(r,r)},$$

$$T_1(r,r) = \begin{pmatrix} \overline{i}_{(p)}, \overline{\overline{j}}_{(rq)}, \overline{k}_{(p)}, \overline{\overline{l}}_{(rq)} \\ \overline{\overline{j}}_{(rq)}, \overline{\overline{l}}_{(rq)}, \overline{i}_{(p)}, \overline{k}_{(p)} \end{pmatrix}, \tag{68}$$

where $\overline{i}_{(p)}$, $\overline{k}_{(p)}$ are the $p$-multiindices; $\overline{\overline{j}}_{(rq)}$, $\overline{\overline{l}}_{(rq)}$ are the sets of the multiindices, consisting of $r$ $q$-multiindices. If the errors $\varepsilon_i$ have the normal distribution $N(0,V_\varepsilon)$, then the estimation $\widehat{\beta}_{(p,rq)}$ has the normal distribution $N(\beta_{p,rq}, V_{\widehat{\beta}_r})$, as it follows from the expressions (37), (38), (44).

If we fix the values of $\overline{i}_{(p)}$ and $\overline{\overline{j}}_{(rq)}$, i. e. $\overline{i}_{(p)} = \overline{i}^*_{(p)}$, $\overline{\overline{j}}_{(rq)} = \overline{\overline{j}}^*_{(rq)}$, then $\widehat{\beta}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}$ will be a separate element of the matrix $\widehat{\beta}_{(p,rq)}$ (30). The mathematical expectation of the estimation $\widehat{\beta}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}$ is equal to $\widehat{\beta}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}$ and the variation of the estimation $\widehat{\beta}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}$ is equal to $v^{(\widehat{\beta}_r)}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)},\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}$. It is clear, that $\widehat{\beta}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}} \in N\left( \beta_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}, v^{(\widehat{\beta}_r)}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)},\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}} \right)$, where $\widehat{\beta}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}$ and $v^{(\widehat{\beta}_r)}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)},\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}$ are defined by the formulae (67), (68).

It is clear, that

$$u^{(\widehat{\beta}_r)}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}} = \frac{\widehat{\beta}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}} - \beta_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}}{\sqrt{v^{(\widehat{\beta}_r)}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)},\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}}} \in N(0,1). \tag{69}$$

Let us continue with the estimation $\widehat{V}_\varepsilon$. Since $s_2 = s_3 = s_4$ in the expression (57), and $s_2$ is defined by the expression (60), then

$$\widehat{V}_\varepsilon = s_1 - s_2, \tag{70}$$

where

$$s_1 = \frac{1}{l} \sum_{i=1}^{l} {}^{0,0}(\varepsilon_i \varepsilon_i), \tag{71}$$

$$s_2 = \sum_{s=0}^{m} \frac{1}{s!} \frac{1}{l^2} \sum_{i=1}^{l} \sum_{j=1}^{l} {}^{0,0}\left( \varepsilon_i {}^{0,0}(m_{s,i,j} \varepsilon_j) \right), \tag{72}$$

$$m_{s,i,j} = {}^{0,sq}\left( Q_s(x_i) P_s(x_j) \right).$$

The expressions (71) and (72) can be represented in the following forms:

$$s_1 = \frac{1}{l} \sum_{i=1}^{l} {}^{0,0}(\varepsilon_i \varepsilon_i) = \frac{1}{l} \sum_{i=1}^{l} \sum_{j=1}^{l} {}^{0,0}\left( \varepsilon_i {}^{0,0}(\delta_{i,j} \varepsilon_j) \right),$$

$$s_2 = \sum_{s=0}^{m} \frac{1}{l} \sum_{i=1}^{l} \sum_{j=1}^{l} {}^{0,0}\left( \varepsilon_i {}^{0,0}(M_{s,i,j} \varepsilon_j) \right) = \frac{1}{l} \sum_{i=1}^{l} \sum_{j=1}^{l} {}^{0,0}\left( \varepsilon_i {}^{0,0}(M_{i,j} \varepsilon_j) \right),$$

where $\delta_{i,j}$ is Kronecker delta and

$$M_{s,i,j} = \frac{m_{s,i,j}}{s!l} = \frac{1}{s!l} {}^{0,sq}\left( Q_s(x_i) P_s(x_j) \right), \tag{73}$$

$$M_{i,j} = \sum_{s=0}^{m} M_{s,i,j} = \sum_{s=0}^{m} \frac{1}{s!l} {}^{0,sq}\left( Q_s(x_i) P_s(x_j) \right). \tag{74}$$

This allows us to represent $\widehat{V}_\varepsilon$ (70) as

$$\widehat{V}_\varepsilon = \frac{1}{l}\sum_{i=1}^{l}\sum_{j=1}^{l} {}^{0,0}\left(\varepsilon_i \,{}^{0,0}(U_{i,j}\varepsilon_j)\right), \tag{75}$$

where $U_{i,j} = \delta_{i,j} - \sum_{s=0}^{m} M_{s,i,j}$. $\widehat{V}_\varepsilon l$ (75) is the quadratic form with the matrix ($U_{i,j}$). Since the matrices ($M_{s,i,j}$) (73) are idempotent (Appendix, theorem 3), the matrix ($M_{i,j}$) (74) is idempotent (Appendix, theorem A4) and the matrix ($U_{i,j}$) is idempotent (Appendix, consequence from theorem A6), then $\widehat{V}_\varepsilon l$ (75) has Wishart distribution $W(k,V_\varepsilon)$ with the degrees of freedom number $k$ which is equal to the trace of the matrix ($U_{i,j}$) [19]. Since $\mathrm{tr}\,(e_{i,j}) = l$, $\mathrm{tr}\,(M_{s,i,j}) = \frac{(n+s-1)!}{s!(n-1)!}$ (Appendix, formula (A6)), then

$$k = \mathrm{tr}\,(U_{i,j}) = \mathrm{tr}\,(e_{i,j}) - \mathrm{tr}\left(\sum_{s=0}^{m}(M_{s,i,j})\right) = l - \sum_{s=0}^{m}\frac{(n+s-1)!}{s!(n-1)!}, \tag{76}$$

where $n = n_1 n_2 \cdots n_q$ and $(n_1, n_2, \cdots, n_q)$ is the size of the $q$-dimensional-matrix argument $x$.

The statistic

$$\chi_k^2 = \frac{{}^{0,p}\left(L\,{}^{0,p}(\widehat{V}_\varepsilon l L)\right)}{{}^{0,p}\left(L\,{}^{0,p}(V_\varepsilon L)\right)} \in H(k), \tag{77}$$

where $\widehat{V}_\varepsilon$ is calculated by the formula (36) for an arbitrary $p$-dimensional matrix $L$ of $\varepsilon$ (or $y$) size (Appendix, theorem A7).

Let us continue with the estimation $\widehat{y}_i$ (52). Since the estimation $\widehat{y}_i$ is the linear function of the normal distributed estimations $\hat{\mathring{\beta}}_{(rq,p)}$, then it has the normal distribution: $\widehat{y}_i \in N(y_i, V_{\widehat{y}_i})$, where $V_{\widehat{y}_i}$ is defined by the formula (54). In such case

$$u^{(y)}_{\bar{i}^*(p)} = \frac{\widehat{y}_{\bar{i}^*(p)} - y_{\bar{i}^*(p)}}{\sqrt{v^{(y)}_{\bar{i}^*(p),\bar{i}^*(p)}}} \in N(0,1). \tag{78}$$

The estimations $\widehat{c}_{(p,rq)}$ of the coefficients $c_{(p,rq)}$ in the mdm representation of the mdm $m$ degree polynomial $y(x)$ (17) (31) or (34) has the normal distribution:

$$\widehat{c}_{(p,rq)} \in N(c_{(p,rq)}, V_{\widehat{c}_r}),$$

where $V_{\widehat{c}_r}$ is defined by the expressions (66) or (65). Then

$$u^{(\widehat{c}_r)}_{\bar{i}^*(p),\bar{\bar{j}}^*(rq)} = \frac{\widehat{c}_{\bar{i}^*(p),\bar{\bar{j}}^*(rq)} - c_{\bar{i}^*(p),\bar{\bar{j}}^*(rq)}}{\sqrt{v^{(\widehat{c}_r,r)}_{\bar{i}^*(p),\bar{\bar{j}}^*(rq),\bar{i}^*(p),\bar{\bar{j}}^*(rq)}}} \in N(0,1). \tag{79}$$

T h e o r e m 7. *Let the errors $\varepsilon_i$ in the measurements model* (24) *have zero mean value* $E(\varepsilon_i) = 0$*, the covariance matrix* $V_\varepsilon = E({}^{0,0}\varepsilon_i^2) = E({}^{0,0}\varepsilon^2)$*, the normal distribution* $N(0,V_\varepsilon)$*, and are independent by i. Then*

*1) the estimation* $\widehat{\beta}_{(p,rq)}$ *has the normal distribution* $N(\beta_{p,rq}, V_{\widehat{\beta}_r})$;

*2) the scalar statistics* $u^{(\widehat{\beta}_r)}_{\bar{i}^*(p),\bar{\bar{j}}^*(rq)}$ *defined by the expression* (69) *have the normal distribution* $N(0,1)$;

*3) the scalar statistic* $\chi_k^2$ *defined by the expression* (77) *has the chi-square distribution with k degrees of freedom;*

*4) the estimation* $\widehat{y}_i$ *of the respond has the normal distribution* $N(y_i, V_{\widehat{y}_i})$;

*5) the scalar statistics* $u^{(y)}_{\bar{i}^*(p)}$ *defined by the expression* (78) *have the normal distribution* $N(0,1)$;

*6) the estimation* $\widehat{c}_{(p,rq)}$ *has the normal distribution* $N(c_{(p,rq)}, V_{\widehat{c}_r})$;

*7) the scalar statistics* $u^{(\widehat{c}_r)}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}$ *defined by the expression* (79) *have the normal distribution N*(0,1).

***2.8. Distributions of the statistics related to the Fourier series.*** The estimation $\mathring{\widehat{\beta}}_{(p,rq)} = \frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i P_r(x_i)\right) = \sum_{i=1}^{l}{}^{0,0}\left(\varepsilon_i P_{r,i}\right)$ is a linear form with the matrix $(P_{r,i}) = \left(P_r(x_i)/l\right)$, and the estimation $\widehat{V}_\varepsilon = \frac{1}{l}\sum_{i=1}^{l}\sum_{j=1}^{l}{}^{0,0}\left(\varepsilon_i\, {}^{0,0}(U_{i,j}\varepsilon_j)\right)$ is a quadratic form with the matrix

$$(U_{i,j}) = \left(\delta_{i,j} - \sum_{s=0}^{m}\frac{1}{s!l}{}^{0,sq}\left(Q_s(x_i)P_s(x_j)\right)\right).$$

The estimations $\mathring{\widehat{\beta}}_{(p,rq)}$ and $\widehat{V}_\varepsilon$ are independent if and only if ${}^{0,1}\left((P_{r,i})(U_{i,j})\right) = 0$ [19]. We have

$$\begin{aligned}
{}^{0,1}\left((P_{r,i})(U_{i,j})\right) &= \left(\sum_{i=1}^{l}{}^{0,0}(P_{r,i}U_{i,j})\right) = \left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)\left(\delta_{i,j} - \sum_{s=0}^{m}\frac{1}{s!l}{}^{0,sq}\left(Q_s(x_i)P_s(x_j)\right)\right)\right)\right) = \\
&= \left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)\delta_{i,j}\right)\right) - \left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)\left(\sum_{s=0}^{m}\frac{1}{s!l}{}^{0,sq}\left(Q_s(x_i)P_s(x_j)\right)\right)\right)\right) = \\
&= \left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)\delta_{i,j}\right)\right) - \left(\sum_{s=0}^{m}\frac{1}{s!}\frac{1}{l}{}^{0,sq}\left(\left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)Q_s(x_i)\right)\right)P_s(x_j)\right)\right).
\end{aligned} \tag{80}$$

The first summand in the expression (80) is equal

$$\left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)\delta_{i,j}\right)\right) = \frac{1}{l}P_r(x_j).$$

Let us find the second summand in (80).

$$\left(\sum_{s=0}^{m}\frac{1}{s!}\frac{1}{l}{}^{0,sq}\left(\left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)Q_s(x_i)\right)\right)P_s(x_j)\right)\right) =$$

$$= \left(\sum_{\substack{s=0\\ s\neq r}}^{m}\frac{1}{s!}\frac{1}{l}{}^{0,sq}\left(\left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)Q_s(x_i)\right)\right)P_s(x_j)\right)\right) + \left(\frac{1}{r!}\frac{1}{l}{}^{0,rq}\left(\left(\frac{1}{l}\sum_{i=1}^{l}{}^{0,0}\left(P_r(x_i)Q_r(x_i)\right)\right)P_r(x_j)\right)\right) = \frac{1}{l}P_r(x_j),$$

due to the condition (16). Thus, ${}^{0,1}\left((P_{r,i})(U_{i,j})\right) = \frac{1}{l}P_r(x_j) - \frac{1}{l}P_r(x_j) = 0,$ and the estimations $\mathring{\widehat{\beta}}_{(p,rq)}$ and $\widehat{V}_\varepsilon$ are independent. Then the statistic

$$t^{(\widehat{\beta}_r)}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}} = \frac{u^{(\widehat{\beta}_r)}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}}{\sqrt{\chi^2_k}}\sqrt{k} \in T(k), \tag{81}$$

where $u^{(\widehat{\beta}_r)}_{\overline{i}^*_{(p)},\overline{\overline{j}}^*_{(rq)}}$ and $\chi^2_k$ are calculated by the formulae (69) and (77) with the use of the arbitrary $L$ and $V_\varepsilon$.

Let us consider the statistic relating to the estimation $\widehat{y}_i$ (52). Since the estimations $\mathring{\widehat{\beta}}_{rq,p}$ in formula (52) and the estimation $\widehat{V}_\varepsilon$ (36) are independent, then the estimations $\widehat{y}_i$ and $\widehat{V}_\varepsilon$ are independent too. In such a case, the statistic

$$t^{(y)}_{\bar{\bar{i}}^*(p)} = \frac{u^{(y)}_{\bar{\bar{i}}^*(p)}}{\sqrt{\chi^2_k}}\sqrt{k} = \frac{\hat{y}_{\bar{\bar{i}}^*(p)} - y_{\bar{\bar{i}}^*(p)}}{\sqrt{v^{(y)}_{\bar{\bar{i}}^*(p),\bar{\bar{i}}^*(p)}}\sqrt{\chi^2_k}}\sqrt{k} \in T(k), \tag{82}$$

where $u^{(y)}_{\bar{\bar{i}}^*(p)}$ and $\chi^2_k$ are calculated by the formulae (78) and (77) with use the arbitrary $L$ and $V_\varepsilon$.

We get analogously that

$$t^{(\hat{c}_r)}_{\bar{\bar{i}}^*(p),\bar{\bar{j}}^*(rq)} = \frac{u^{(\hat{c}_r)}_{\bar{\bar{i}}^*(p),\bar{\bar{j}}^*(rq)}}{\sqrt{\chi^2_k}}\sqrt{k} \in T(k), \tag{83}$$

where $u^{(\hat{c}_r)}_{\bar{\bar{i}}^*(p),\bar{\bar{j}}^*(rq)}$ and $\chi^2_k$ are calculated by the formulae (79) and (77).

The statistics $t^{(\beta_r)}_{\bar{\bar{i}}^*(p),\bar{\bar{j}}^*(rq)}$ (81), $t^{(y)}_{\bar{\bar{i}}^*(p)}$ (82) and $t^{(\hat{c}_r)}_{\bar{\bar{i}}^*(p),\bar{\bar{j}}^*(rq)}$ (83) can be used for the hypothesis testing relating to $\beta_{(p,rq)}$, $y$ and $c_{(p,rq)}$.

T h e o r e m 8. *Under the conditions of Theorem 7*

*1) the estimations* $\hat{\beta}_{(p,rq)}$ (30) *and* $\hat{V}_\varepsilon$ (36) *are independent;*

*2) the scalar statistics* $t^{(\hat{\beta}_r)}_{\bar{\bar{i}}^*(p),\bar{\bar{j}}^*(rq)}$ *defined by the expression* (81) *have Student distribution with k degrees of freedom;*

*3) the scalar statistic* $t^{(y)}_{\bar{\bar{i}}^*(p)}$ *defined by the expression* (82) *has Student distribution with k degrees of freedom;*

*5) the scalar statistics* $t^{(\hat{c}_r)}_{\bar{\bar{i}}^*(p),\bar{\bar{j}}^*(rq)}$ *defined by the expression* (83) *have Student distribution with k degrees of freedom.*

***2.9. Computer simulation of the regression analysis.*** Let us consider the quiet simple regression model

$$y_o = {}^{0,0q}\left(c_{(p,0q)}x^0\right) + {}^{0,q}\left(c_{(p,q)}x^1\right) + {}^{0,2q}\left(c_{(p,2q)}x^2\right) + {}^{0,3q}\left(c_{(p,3q)}x^3\right) + \varepsilon \tag{84}$$

with $p=1,\ q=1,\ y_o=(y_{o,i}),\ x=(x_i),\ \varepsilon=(\varepsilon_i),\ i=1,2,$ and with the following coefficients:

$$c_{(p,0q)} = (c_i) = (5 \quad 3), \quad c_{(p,q)} = (c_{i,j}) = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix},$$

$$c_{(p,2q)} = (c_{i,j_1,j_2}) = \begin{matrix} \\ (1) \\ (2) \end{matrix}\begin{matrix} (1,1) & (2,1) & (1,2) & (2,2) \\ c_{111} & c_{121} & c_{112} & c_{122} \\ c_{211} & c_{221} & c_{212} & c_{222} \end{matrix} = \begin{matrix} \\ (1) \\ (2) \end{matrix}\begin{matrix} (1,1) & (2,1) & (1,2) & (2,2) \\ 0 & 0 & 0 & 4 \\ 2 & 2.5 & 2.5 & 5 \end{matrix},$$

$$c_{(p,3q)} = (c_{i,j_1,j_2,j_3}) = \begin{matrix} \\ (1) \\ (2) \end{matrix}\begin{matrix} (1,1,1) & (1,2,1) & (1,1,2) & (1,2,2) & (2,1,1) & (2,2,1) & (2,1,2) & (2,2,2) \\ c_{1111} & c_{1121} & c_{1112} & c_{1122} & c_{1211} & c_{1221} & c_{1212} & c_{1222} \\ c_{2111} & c_{2121} & c_{2112} & c_{2122} & c_{2211} & c_{2221} & c_{2212} & c_{2222} \end{matrix} =$$

$$= \begin{matrix} \\ (1) \\ (2) \end{matrix}\begin{matrix} (1,1,1) & (1,2,1) & (1,1,2) & (1,2,2) & (2,1,1) & (2,2,1) & (2,1,2) & (2,2,2) \\ 4 & 0.3 & 0.3 & -2.3 & 0.3 & -2.3 & -2.3 & 2 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{matrix}.$$

We can see that some elements of these matrices are equal to zero. They are $c_{11}$, $c_{111}$, $c_{121}$, $c_{112}$, $c_{2121}$, $c_{2112}$, $c_{2122}$, $c_{2211}$, $c_{2221}$, $c_{2212}$. The covariance matrix of the measurements error ε is chosen to be equal

$$V_\varepsilon = (v_{i,j}) = \begin{pmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{pmatrix}.$$

The design of the experiment is simulated as follows: each component of one-dimensional matrix $x$ takes 4 values linearly spaced between and including –1, 1, in sum 16 measurements. The estimations

$\widehat{c}_{(p,rq)}$ of the coefficients $c_{(p,rq)}$ in (84) were calculated by the formula (34). The real function (the polynomial in (84)) and its orthogonal approximation (the orthogonal regression function)

$$\widehat{y} = {}^{0,0q}\left(\widehat{c}_{(p,0q)}x^0\right) + {}^{0,q}\left(\widehat{c}_{(p,q)}x^1\right) + {}^{0,2q}\left(\widehat{c}_{(p,2q)}x^2\right) + {}^{0,3q}\left(\widehat{c}_{(p,3q)}x^3\right), \tag{85}$$

were calculated with use of the mdm Horner scheme [20]. The confidence interval for the response was calculated on the base of the $t$-statistics $t^{(y)}_{\overline{i}^*_{(p)}}$ (82) for the confidence probability $\gamma = 0.95$. It looks as follows:

$$\widehat{y}_{\overline{i}^*_{(p)}} - \frac{t_{(1-\gamma)/2}}{\sqrt{k}}\sqrt{v^{(y)}_{\overline{i}^*_{(p)},\overline{i}^*_{(p)}}}\sqrt{\chi^2_k} \le y_{\overline{i}^*_{(p)}} \le \widehat{y}_{\overline{i}^*_{(p)}} + \frac{t_{(1-\gamma)/2}}{\sqrt{k}}\sqrt{v^{(y)}_{\overline{i}^*_{(p)},\overline{i}^*_{(p)}}}\sqrt{\chi^2_k}, \tag{86}$$

where $t_{(1-\gamma)/2}$ is $100(1-\gamma)/2$ percent point of $t$ (Student) distribution with $k$ degreases of freedom; $\widehat{y}_{\overline{i}^*_{(p)}}$ is $\overline{i}^*_{(p)}$-th element of the matrix $\widehat{y} = (\widehat{y}_{\overline{i}_{(p)}})$ (85); $v^{(y)}_{\overline{i}^*_{(p)},\overline{i}^*_{(p)}}$ is $(\overline{i}^*_{(p)},\overline{i}^*_{(p)})$-th element of the matrix $V_{\widehat{y}} = \left(v^{(y)}_{\overline{i}_{(p)},\overline{j}_{(p)}}\right)$ (54); $\chi^2_k$ is the statistic (77) and $k$ is the parameter (76). Figure 1 shows the surfaces of the real regression function $y_1 = y_1(x_1,x_2)$ and its omdm-approximation $\widehat{y}_1 = y_1(x_1,x_2)$. Figure 2 shows the surfaces of the real regression function $y_1 = y_1(x_1,x_2)$ and its confidence interval (86).

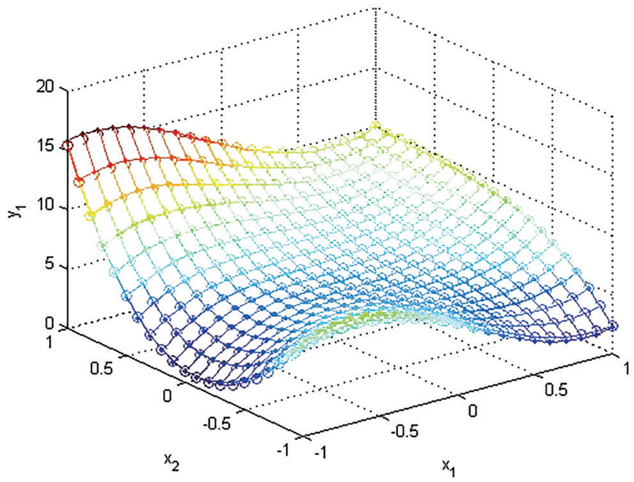


Fig. 1. The surfaces of the real regression function $y_1 = y_1(x_1,x_2)$
and its omdm-approximation $\widehat{y}_1 = y_1(x_1,x_2)$: (·) – real function; (o) – omdm-approximation

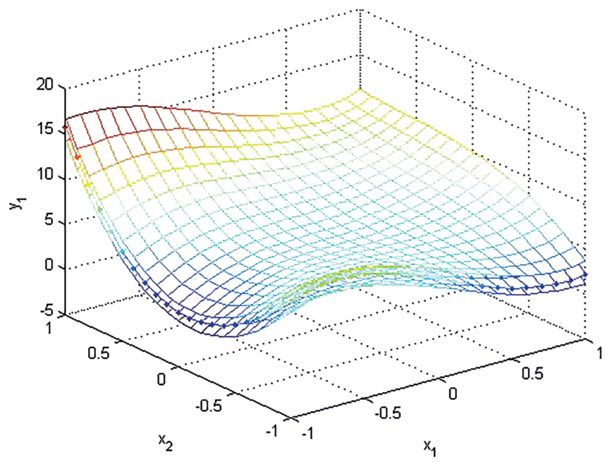


Fig. 2. The surfaces of the real regression function $y_1 = y_1(x_1,x_2)$ and its confidence interval (86): (·) – real function

The hypothesis testing relating to the coefficients $c_{(p,rq)}$ in (84) was performed on the base of *t*-statistics (83) at the significance level α = 0.05. All of zero coefficients were recognized as insignificant, and all of non-zero coefficients were recognized as significant.

**Conclusion.** The known results of Fourier series by the orthogonal polynomials are extended to the case of the mdm functions and mdm arguments. The analytical expressions for the orthogonal polynomials and Fourier series of the second degree useful for the potential analytical studies are obtained. The theoretical results are realized as the single program function for any cases. This software feature we call the algorithmic generality. The efficiency of the program function is confirmed on the instance, performing of which is impossible by the classical approach. The theory of the orthogonal mdm regression analysis is developed and confirmed by the computer simulation.

**Appendix.** Theorem A1. *The following condition for the orthogonal polynomials* $Q_r(x)$ *and* $P_r(x)$ *hold:*

$$\int\limits_{\Omega} {}^{0,rq}\left(Q_r(x)P_r(x)\right)\rho(x)dx = r!\,\mathrm{tr}\left(E^{(s)}(0,rq)\right), \tag{A1}$$

*where x is the q-dimensional-matrix of the size* $(n_1,n_2,\ldots,n_q)$ *argument;* $E^{(s)}(0,rq)$ *is the symmetrical* $(0,rq)$*-identity of the order* $n = n_1 \cdot n_2 \cdot \ldots \cdot n_q$ *matrix* $E^{(s)}(0,rq)$, $\mathrm{tr}\left(E^{(s)}(0,rq)\right)$ *is the trace of the matrix* $E^{(s)}(0,rq)$. *The equation* (A1) *for the sample distribution as the weight function seems as follows:*

$$\frac{1}{l}\sum_{i=1}^{l} {}^{0,rq}\left(Q_r(x_i)P_r(x_i)\right) = r!\,\mathrm{tr}\left(E^{(s)}(0,rq)\right). \tag{A2}$$

Proof. Let $q_{\overline{\overline{c}}}(x)$, $p_{\overline{\overline{m}}}(x)$ and $e^{(s)}_{\overline{\overline{c}},\overline{\overline{m}}}$ be the elements of the matrices $Q_r(x)$, $P_r(x)$ and $E^{(s)}(0,rq)$ respectively, i. e. $Q_r(x) = \left(q_{\overline{\overline{c}}}(x)\right)$, $P_r(x) = \left(p_{\overline{\overline{m}}}(x)\right)$, $E^{(s)}(0,rq) = \left(e^{(s)}_{\overline{\overline{c}},\overline{\overline{m}}}\right)$. Then

$${}^{0,rq}\left(Q_r(x)P_r(x)\right) = \sum_{\overline{\overline{c}}} q_{\overline{\overline{c}}}(x)p_{\overline{\overline{c}}}(x). \tag{A3}$$

Integration of the expression (A3) with the weight ρ(*x*) gives

$$\int\limits_{\Omega} {}^{0,r}\left(Q_r(x)P_r(x)\right)\rho(x)dx = \sum_{\overline{\overline{c}}}\int\limits_{\Omega} q_{\overline{\overline{c}}}(x)p_{\overline{\overline{c}}}(x)\rho(x)dx. \tag{A4}$$

The orthogonality condition (5) means that

$$\int\limits_{\Omega} q_{\overline{\overline{c}}}(x)p_{\overline{\overline{m}}}(x)\rho(x)dx = r!\,e^{(s)}_{\overline{\overline{c}},\overline{\overline{m}}}$$

and

$$\int\limits_{\Omega} q_{\overline{\overline{c}}}(x)p_{\overline{\overline{c}}}(x)\rho(x)dx = r!\,e^{(s)}_{\overline{\overline{c}},\overline{\overline{c}}}. \tag{A5}$$

Substitution (A5) into (A4) gives

$$\int\limits_{\Omega} {}^{0,r}\left(Q_r(x)P_r(x)\right)\rho(x)dx = r!\sum_{\overline{\overline{c}}} e^{(s)}_{\overline{\overline{c}},\overline{\overline{c}}}.$$

Since the element $e^{(s)}_{\overline{\overline{c}},\overline{\overline{c}}}$ is the diagonal element of the matrix $E^{(s)}(0,rq)$ $(rq,0,rq)$-associated with the matrix $E^{(s)}(0,rq)$, then $\sum_{\overline{\overline{c}}} e^{(s)}_{\overline{\overline{c}},\overline{\overline{c}}}$ is the trace of the matrix $E^{(s)}(0,rq)$, and the equation (A1) holds. Theorem A1 is proved.

Theorem A2. *The trace of the matrix* $E^{(s)}(0,rq)$ $(rq,0,rq)$*-associated with the symmetrical* $(0,rq)$*-identity of the order* $n = n_1 \cdot n_2 \cdot \ldots \cdot n_q$ *matrix* $E^{(s)}(0,rq)$ *is defined by the following expression:*

$$\mathrm{tr}\left(E^{(s)}(0,rq)\right) = C^r_{n+r-1} = \frac{(n+r-1)!}{r!(n-1)!}, \tag{A6}$$

*where* $(n_1,n_2,\ldots,n_q)$ *is the size of the multiindex q.*

P r o o f. We need the following definition of the symmetrical $(0,rq)$-identity of the order $n = n_1 \cdot n_2 \cdot ... \cdot n_q$ matrix $E^{(s)}(0,rq)$:

$$E^{(s)}(0,rq) = \left(e^{(s)}_{\bar{\bar{c}},\bar{\bar{m}}}\right) = \left(\begin{cases} \dfrac{\mu_1!\mu_2!...\mu_n!}{\mu!}, & \text{if } \operatorname{perm}(\bar{\bar{c}}) = \bar{\bar{m}}, \\ 0, & \text{if } \operatorname{perm}(\bar{\bar{c}}) \neq \bar{\bar{m}} \end{cases}\right), \tag{A7}$$

where $\mu_i$ is the number of repetitions of the number $i$ in the set of the multiidices $\bar{\bar{c}}$ (or $\bar{\bar{m}}$), $i = 1,2,...,n$; $n = n_1 \cdot n_2 \cdot ... \cdot n_q$ is the power of the multiindex $q$; $(n_1, n_2, ..., n_q)$ is the size of the multiindex $q$; $\mu_1 + \mu_2 + ... + \mu_n = r$; $r$ is the number of the multiindices in the set of the multiidices $\bar{\bar{c}}$; $\operatorname{perm}(\bar{\bar{c}})$ is some permute of the multiindices in $\bar{\bar{c}}$.

The diagonal element $e^{(s)}_{\bar{\bar{c}},\bar{\bar{c}}}$ of the matrix $E^{(s)}(0,rq)$ has, in accordance with the definition (A7), the value $e^{(s)}_{\bar{\bar{c}},\bar{\bar{c}}} = \dfrac{\mu_1!\mu_2!...\mu_n!}{r!}$ and repeats on the diagonal $\dfrac{r!}{\mu_1!\mu_2!...\mu_n!}$ times, which gives one in sum. Since each element with the structure $\mu_1, \mu_2, ..., \mu_n$ lies on the diagonal, and the contribution of each of them in trace is equal to one, then the trace of the matrix is equal to the number of the structures $C^r_{n+r-1}$. So we have $\sum_{\bar{\bar{c}}} e^{(s)}_{\bar{\bar{c}},\bar{\bar{c}}} = \operatorname{tr}\left(\tilde{E}^{(s)}(0,rq)\right) = C^r_{n+r-1} = \dfrac{(n+r-1)!}{r!(n-1)!}$. Theorem A2 is proved.

T h e o r e m A3. *The matrix*

$$M_r = (M_{r,i,j}) = \left(\frac{m_{r,i,j}}{r!l}\right) = \left(\frac{1}{r!l}\,{}^{0,rq}\left(Q_r(x_i)P_r(x_j)\right)\right) \tag{A8}$$

*is idempotent. Its trace is equal:*

$$\operatorname{tr}(M_r) = \operatorname{tr}\left(E^{(s)}(0,rq)\right) = C^r_{n+r-1} = \frac{(n+r-1)!}{r!(n-1)!}. \tag{A9}$$

P r o o f. Let us find the matrix $M2_r = M_r \cdot M_r$:

$$M2_r = M_r \cdot M_r = \sum_{\nu=1}^{l} M_{r,i,\nu} M_{r,\nu,j} = \frac{1}{(r!l)^2} \sum_{\nu=1}^{l}{}^{0,0}\left({}^{0,rq}\left(Q_r(x_i)P_r(x_\nu)\right){}^{0,rq}\left(Q_r(x_\nu)P_r(x_j)\right)\right) =$$

$$= \frac{1}{(r!l)^2}\,{}^{0,rq}\left(Q_r(x_i)\,{}^{0,rq}\left(\sum_{\nu=1}^{l}{}^{0,0}\left(P_r(x_\nu)Q_r(x_\nu)\right)P_r(x_j)\right)\right).$$

Since

$$\sum_{i=1}^{l}{}^{0,0}\left(Q_r(x_i)P_r(x_i)\right) = r!lE^{(s)}(0,rq),$$

then

$$M2_r = \frac{1}{(r!l)}\,{}^{0,rq}\left(Q_r(x_i)\,{}^{0,rq}\left(E^{(s)}(0,rq)P_r(x_j)\right)\right) = \frac{1}{(r!l)}\,{}^{0,rq}\left(Q_r(x_i)P_r(x_j)\right) = M_r.$$

The equation (A9) follows from the theorems A1, A2 of the Appendix. Theorem A3 is proved.

T h e o r e m A4. *If $r \neq s$, then* ${}^{0,1}(M_r M_s) = 0$.

P r o o f.

$${}^{0,1}(M_r M_s) = \sum_{\nu=1}^{l} M_{r,i,\nu} M_{s,\nu,j} = \frac{1}{(r!l)(s!l)} \sum_{\nu=1}^{l}{}^{0,0}\left({}^{0,rq}\left(Q_r(x_i)P_s(x_\nu)\right){}^{0,rq}\left(Q_r(x_\nu)P_s(x_j)\right)\right) =$$

$$= \frac{1}{(r!l)(s!l)}\,{}^{0,rq}\left(Q_r(x_i)\,{}^{0,rq}\left(\sum_{\nu=1}^{l}{}^{0,0}\left(P_s(x_\nu)Q_r(x_\nu)\right)P_s(x_j)\right)\right) = 0$$

since $\sum_{\nu=1}^{l}{}^{0,0}\left(P_s(x_\nu)Q_r(x_\nu)\right) = 0$. Theorem A4 is proved.

T h e o r e m A5. *The matrix*

$$M = \sum_{r=0}^{m} M_r, \tag{A10}$$

*where* $M_r$ *is defined by the formula* (A8)*, is idempotent.*

P r o o f. We have on the base of the theorems A4, A3 the following equations:

$$^{0,1}(MM) = {}^{0,1}\left(\left(\sum_{r=0}^{m} M_r\right)\left(\sum_{s=0}^{m} M_s\right)\right) = \sum_{r=0}^{m}\sum_{s=0}^{m} {}^{0,1}(M_r M_s) = \sum_{r=0}^{m} {}^{0,1}(M_r M_r) = \sum_{r=0}^{m} M_r = M.$$

Theorem A5 is proved.

T h e o r e m A6. *If the order l matrix A is idempotent and I is the order l identity matrix, then the matrix I – A is idempotent.*

P r o o f. This known result is proved as follows:

$$(I - A)(I - A) = I - A - A + A \cdot A = I - A - A + A = I - A.$$

Theorem A6 is proved.

C o n s e q u e n c e. If $M$ is the matrix (A10) and $I$ is the order $l$ identity matrix, then the matrix $I - M$ is idempotent.

D e f i n i t i o n. Let $\xi = (\xi_{i_1, i_2, \ldots, i_p})$ be a $p$-dimensional random matrix with normal distribution $N(0,V)$, $x_1, x_2, \ldots, x_l$ is the random sample of the volume $l$ from this distribution and $W = \sum_{i=1}^{l} {}^{0,0}x_i^2$. We will call the distribution of the random matrix $W$ as the central Wishart distribution with $l$ degreases of freedom and parametric matrix $V$ and will denote such a distribution $W(l,V)$ and will write $W \in W(l,V)$.

T h e o r e m A7. *If* 2*p-dimensional random matrix W has Wishart distribution W*(*l*,*V*) $\left(W \in W(l,V)\right)$, *and L is the fixed arbitrary p-dimensional matrix admitting the product* $^{0,p}\left(L\, ^{0,p}(WL)\right)$, *then a random variable* $\chi^2 = \frac{^{0,p}\left(L\, ^{0,p}(WL)\right)}{^{0,p}\left(L\, ^{0,p}(VL)\right)}$ *has the* $\chi^2$ *distribution with l degreases of freedom* $H_l$*:*

$$\chi^2 = \frac{^{0,p}\left(L\, ^{0,p}(WL)\right)}{^{0,p}\left(L\, ^{0,p}(VL)\right)} \in H_l.$$

P r o o f. The matrix $W$ can be represented in the following form: $W = \sum_{i=1}^{l} {}^{0,0}(U_i U_i)$, where $U_i \in N(0,V)$, and the random $p$-dimensional matrices $U_1, U_2, \ldots, U_l$ are independent. We have

$$^{0,p}\left(L\, ^{0,p}(WL)\right) = {}^{0,p}\left(L\, ^{0,p}\left(\sum_{i=1}^{l} {}^{0,0}(U_i U_i) L\right)\right) = \sum_{i=1}^{l} {}^{0,p}\left(L\, ^{0,p}\left(^{0,0}(U_i U_i) L\right)\right) = \sum_{i=1}^{l} {}^{0,0}\left(^{0,p}(LU_i)\right)^2.$$

The random variables $^{0,p}(LU_i)$ are independent on *i*, $^{0,p}(LU_i) \in N\left(0, {}^{0,p}\left(L\, ^{0,p}(VL)\right)\right)$ and the variable $\chi^2 = \frac{^{0,p}\left(L\, ^{0,p}(WL)\right)}{^{0,p}\left(L\, ^{0,p}(VL)\right)}$ satisfies the definition of $\chi^2$ distribution with $l$ degreases of freedom. Theorem A7 is proved.

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## Information about the author

**Vladimir S. Mukha** – Dr. Sc. (Engineering), Professor, Professor of the Department of Information Technologies of Automated Systems, Belarusian State University of Informatics and Radioelectronics (6, P. Brovka Str., 220013, Minsk, Republic of Belarus). E-mail: mukha@bsuir.by

## Информация об авторе

**Муха Владимир Степанович** – доктор технических наук, профессор, профессор кафедры информационных технологий автоматизированных систем, Белорусский государственный университет информатики и радиоэлектроники (ул. П. Бровки, 6, 220013, Минск, Республика Беларусь). E-mail: mukha@bsuir.by