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EIGENVALUES OF THE GENERALIZED HELICITY OPERATOR FOR SPIN 3/2 PARTICLE IN THE PRESENCE OF THE MAGNETIC FIELD AND THE PROJECTIVE OPERATORS METHOD

Abstract. The eigenvalue problem for generalized helicity operator for a spin 3/2 particle in presence of the uniform magnetic field is solved. After separating the variables in the basis of cylindrical coordinates (r, ϕ, z) and the tetrad, the system of 16 first-order differential equations in the variable r is derived. This system is studied with the use of the method of projective operators, constructed with the use of the third projection of the spin for the particle. In accordance with the method by Fedorov – Gronskiy, all 16 variables may be expressed in terms of only 4 distinguished functions, which are constructed in terms of confluent hypergeometric functions. Further the problem reduces to studying the linear algebraic homogeneous system for 16 algebraic variables. In the end, we derive algebraic equations of the second and the fourth order, their roots determine the possible eigenvalues of the helicity operator.

Keywords: spin 3/2 particle, external magnetic field, generalized helicity operator, cylindric symmetry, projective operators, eigenvalue problem, exact solutions

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ЗАДАЧА О СОБСТВЕННЫХ ЗНАЧЕНИЯХ ОБОБЩЕННОГО ОПЕРАТОРА СПИРАЛЬНОСТИ ДЛЯ ЧАСТИЦЫ СО СПИНОМ 3/2 В МАГНИТНОМ ПОЛЕ И МЕТОД ПРОЕКТИВНЫХ ОПЕРАТОРОВ

Аннотация. Решена задача о собственных значениях обобщенного оператора спиральности для частицы со спином 3/2 во внешнем однородном магнитном поле. После разделения переменных в уравнении на собственные значения в цилиндрической системе координат (r, ϕ, z) и соответствующей тетраде найдена система дифференциальных уравнений первого порядка в переменой r для 16 функций. Эта система решена на основе применения метода проективных операторов, построенных на основе третьей проекции оператора спина частицы. В соответствии с методом Федорова – Гронского все 16 переменных могут быть выражены только через 4 различающиеся функции, удовлетворяющие уравнениям вырожденного гипергеометрического типа. Дальнейшая задача сводится к анализу однородной алгебраической системы уравнений для 16 неизвестных величин. В итоге найдены уравнения 2-го и 4-го порядков, корни которых определяют собственные значения оператора спиральности.

Ключевые слова: частица со спином 3/2, внешнее магнитное поле, обобщенный оператор спиральности, цилиндрическая симметрия, проективные операторы, задача на собственные значения, точные решения

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Helicity operator, the basic formulas. As it is known, in presence of the external magnetic field, it is useful to use the possibilities to diagonalize additionally the helicity operator. In Cartesian basis, this operator for a spin 3/2 particle is determined by the formula [1–9] (the presence of the external magnetic field will be taken into account below)

$$i\Big[\left(\partial_1\sigma^{23}+\partial_2\sigma^{31}+\partial_3\sigma^{12}\right)\otimes I+I\otimes\left(\partial_1 j^{23}+\partial_2 j^{31}+\partial_3 j^{12}\right)\Big]=$$
$$=(S_1\partial_1+S_2\partial_2+S_3\partial_3),\quad \Sigma\Psi_{\text{cart}}=\sigma\Psi_{\text{cart}}. \tag{1}$$

Taking in mind expessions for matrices in Cartesian basis, from the formula (1) we derive[1]:

$$\Sigma=\frac{1}{2}\begin{vmatrix}\partial_3 & \partial_1-i\partial_2 & 0 & 0\\ \partial_1+i\partial_2 & -\partial_3 & 0 & 0\\ 0 & 0 & \partial_3 & \partial_1-i\partial_2\\ 0 & 0 & \partial_1+i\partial_2 & -\partial_3\end{vmatrix}\otimes I+I\otimes\begin{vmatrix}0 & 0 & 0 & 0\\ 0 & 0 & -i\partial_3 & i\partial_2\\ 0 & i\partial_3 & 0 & -i\partial_1\\ 0 & -i\partial_2 & i\partial_1 & 0\end{vmatrix}. \tag{2}$$

After the transition to cylindric coordinateds we get

$$\Sigma_{\text{cart}}=\frac{1}{2}\begin{vmatrix}\partial_3 & e^{-i\phi}\left(\partial_r-\frac{i}{r}\partial_\phi\right) & 0 & 0\\ e^{i\phi}\left(\partial_r+\frac{i}{r}\partial_\phi\right) & -\partial_3 & 0 & 0\\ 0 & 0 & \partial_3 & e^{-i\phi}\left(\partial_r-\frac{i}{r}\partial_\phi\right)\\ 0 & 0 & e^{i\phi}\left(\partial_r+\frac{i}{r}\partial_\phi\right) & -\partial_3\end{vmatrix}\otimes I+$$
$$+I\otimes\begin{vmatrix}0 & 0 & 0 & 0\\ 0 & 0 & -i\partial_3 & i\left(\sin\phi\partial_r+\frac{\cos\phi}{r}\partial_\phi\right)\\ 0 & i\partial_3 & 0 & -i\left(\cos\phi\partial_r-\frac{\sin\phi}{r}\partial_\phi\right)\\ 0 & -i\left(\sin\phi\partial_r+\frac{\cos\phi}{r}\partial_\phi\right) & i\left(\cos\phi\partial_r-\frac{\sin\phi}{r}\partial_\phi\right) & 0\end{vmatrix}.$$

Transition to 16-dimensional form. Let us start with the transformation relating Cartesian basis with cylindrical one

$$\Psi_{\text{cyl}}=(B\otimes L)\Psi_{\text{cart}}=\begin{vmatrix}e^{i\phi/2} & e^{i\phi/2}(\cos\phi f_1+\sin\phi f_2) & e^{i\phi/2}(-\sin\phi f_1+\cos\phi f_2) & e^{i\phi/2}f_3\\ e^{-i\phi/2} & e^{-i\phi/2}(\cos\phi g_1+\sin\phi g_2) & e^{-i\phi/2}(-\sin\phi g_1+\cos\phi g_2) & e^{-i\phi/2}g_3\\ e^{i\phi/2} & e^{i\phi/2}(\cos\phi h_1+\sin\phi h_2) & e^{i\phi/2}(-\sin\phi h_1+\cos\phi h_2) & e^{i\phi/2}h_3\\ e^{-i\phi/2} & e^{-i\phi/2}(\cos\phi d_1+\sin\phi d_2) & e^{-i\phi/2}(-\sin\phi d_1+\cos\phi d_2) & e^{-i\phi/2}d_3\end{vmatrix}. \tag{3}$$

We turn to the helicity equation in Cartesian basis; after performing needed calculations we arrive at the system (in Cartesian basis)

$$+\partial_3 f_0^c+e^{-i\phi}\left(\partial_r-\frac{i\partial_\phi}{r}\right)g_0^c=2\sigma f_0^c,\quad -\partial_3 g_0^c+e^{+i\phi}\left(\partial_r+\frac{i\partial_\phi}{r}\right)f_0^c=2\sigma g_0^c,$$
$$+\partial_3 h_0^c+e^{-i\phi}\left(\partial_r-\frac{i\partial_\phi}{r}\right)d_0^c=2\sigma h_0^c,\quad \partial_3 d_0^c+e^{+i\phi}\left(\partial_r+\frac{i\partial_\phi}{r}\right)h_0^c=2\sigma d_0^c;$$

[1] The use of anti-Hermitian generators means that the eigenvalues of the introduced helicity operator will be purely imaginary.

$$+\partial_3 f_1^c - 2i\partial_3 f_2^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)g_1^c + 2i\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)f_3^c = 2\sigma f_1^c,$$

$$\partial_3 g_1^c - 2i\partial_3 g_2^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)f_1^c + 2i\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)g_3^c = 2\sigma g_1^c,$$

$$+\partial_3 h_1^c - 2i\partial_3 h_2^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)d_1^c + 2i\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)h_3^c = 2\sigma h_1^c,$$

$$-\partial_3 d_1^c - 2i\partial_3 d_2^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)h_1^c + 2i\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)d_3^c = 2\sigma d_1^c;$$

$$+\partial_3 f_2^c + 2i\partial_3 f_1^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)g_2^c - 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)f_3^c = 2\sigma f_2^c,$$

$$-\partial_3 g_2^c + 2i\partial_3 g_1^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)f_2^c - 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)g_3^c = 2\sigma g_2^c,$$

$$+\partial_3 h_2^c + 2i\partial_3 h_1^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)d_2^c - 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)h_3^c = 2\sigma h_2^c,$$

$$-\partial_3 d_2^c + 2i\partial_3 d_1^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)h_2^c - 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)d_3^c = 2\sigma d_2^c;$$

$$+\partial_3 f_3^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)g_3^c - 2\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)f_1^c + 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)f_2^c = 2\sigma f_3^c,$$

$$-\partial_3 g_3^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)f_3^c - 2\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)g_1^c + 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)g_2^c = 2\sigma g_3^c,$$

$$+\partial_3 h_3^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)d_3^c - 2\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)h_1^c + 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)h_2^c = 2\sigma h_3^c,$$

$$-\partial_3 d_3^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)h_3^c - 2\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)d_1^c + 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)d_2^c = 2\sigma d_3^c.$$

We can see that it is enough to study only the system for the variables f_a, g_a:

$$+\partial_3 f_0^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)g_0^c = 2\sigma f_0^c, \qquad -\partial_3 g_0^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)f_0^c = 2\sigma g_0^c,$$

$$+\partial_3 f_1^c - 2i\partial_3 f_2^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)g_1^c + 2i\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)f_3^c = 2\sigma f_1^c,$$

$$\partial_3 g_1^c - 2i\partial_3 g_2^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)f_1^c + 2i\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)g_3^c = 2\sigma g_1^c,$$

$$+\partial_3 f_2^c + 2i\partial_3 f_1^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)g_2^c - 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)f_3^c = 2\sigma f_2^c, \tag{4}$$

$$-\partial_3 g_2^c + 2i\partial_3 g_1^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)f_2^c - 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)g_3^c = 2\sigma g_2^c,$$

$$+\partial_3 f_3^c + e^{-i\phi}\left(\partial_r - \frac{i\partial_\phi}{r}\right)g_3^c - 2\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)f_1^c + 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)f_2^c = 2\sigma f_3^c,$$

$$-\partial_3 g_3^c + e^{+i\phi}\left(\partial_r + \frac{i\partial_\phi}{r}\right)f_3^c - 2\left(\sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi\right)g_1^c + 2i\left(\cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi\right)g_2^c = 2\sigma g_3^c.$$

Now we should take into account the transformation between Cartesian and cylindrical tetrad bases; we will follow only transformations on the variables f_a, g_a:

$$\begin{gathered}
f_0^c = e^{i(-1/2)\phi} f_0, \qquad f_1^c = e^{i(-1/2)\phi}\left(\cos\phi f_1 - \sin\phi f_2\right),\\
g_0^c = e^{i(+1/2)\phi} g_0, \qquad g_1^c = e^{i(+1/2)\phi}\left(\cos\phi g_1 - \sin\phi g_2\right),\\
f_2^c = e^{i(-1/2)\phi}\left(\sin\phi f_1 + \cos\phi f_2\right), \qquad f_3^c = e^{i(-1/2)\phi} f_3,\\
g_2^c = e^{i(+1/2)\phi}\left(\sin\phi g_1 + \cos\phi g_2\right), \qquad g_3^c = e^{i(+1/2)\phi} g_3,
\end{gathered} \tag{5}$$

and inverse ones

$$\begin{gathered}
f_0 = e^{i(+1/2)\phi} f_0^c, \qquad f_1 = e^{i(+1/2)\phi}\left(\cos\phi f_1^c + \sin\phi f_2^c\right),\\
g_0 = e^{i(-1/2)\phi} g_0^c, \qquad g_1 = e^{i(-1/2)\phi}\left(\cos\phi g_1^c + \sin\phi g_2^c\right),\\
f_2 = e^{i(+1/2)\phi}\left(-\sin\phi f_1^c + \cos\phi f_2^c\right), \qquad f_3 = e^{i(+1/2)\phi} f_3^c,\\
g_2 = e^{i(-1/2)\phi}\left(-\sin\phi g_1^c + \cos\phi g_2^c\right), \qquad g_3 = e^{i(-1/2)\phi} g_3^c.
\end{gathered} \tag{6}$$

With the use of (5) and (6) we can transform the subsystem (4) to the other form (we will omit technical details and write down the final system of 8 equations; besides, we have taken into account that factors $e^{ikz}e^{im\phi}$ are in the field function)

$$g_0' + \frac{m+1/2}{r} g_0 + f_0(-2\sigma + ik) = 0, \qquad f_0' - \frac{m-1/2}{r} f_0 + g_0(-2\sigma - ik) = 0.$$

$$\begin{gathered}
f_1(-2\sigma + ik) + 2kf_2 + \frac{(1-2m)f_3}{r} + \frac{(2m+1)g_1}{2r} + g_1' + \frac{ig_2}{r} = 0,\\
f_2(-2\sigma + ik) - 2kf_1 - 2if_3' + \frac{(2m+1)g_2}{2r} + g_2' - \frac{ig_1}{r} = 0,\\
\frac{(1-2m)f_1}{2r} + f_1' - \frac{if_2}{r} + g_1(-2\sigma - ik) + 2kg_2 - \frac{(2m+1)g_3}{r} = 0,\\
\frac{(1-2m)f_2}{2r} + f_2' + \frac{if_1}{r} + g_2(-2\sigma - ik) - 2kg_1 - 2ig_3' = 0,\\
(-2\sigma + ik)f_3 + \frac{2m-1}{r} f_1 + 2if_2' + \frac{2i}{r} f_2 + \frac{2m+1}{2r} g_3 + g_3' = 0,\\
\frac{(1-2m)}{2r} f_3 + f_3' + (-2\sigma - ik)g_3 + \frac{2m+1}{r} g_1 + 2ig_2' + \frac{2i}{r} g_2 = 0.
\end{gathered}$$

Thus we arrive at the system of 8 equations. Further we will present this system in the matrix form, applying the truncated column Ψ_{cul}:

$$A\Psi_{\text{cyl}} = 2\sigma \Psi_{\text{cyl}}, \qquad \Psi_{\text{cyl}} = \begin{vmatrix} f_0 \\ g_0 \\ f_1 \\ g_1 \\ f_2 \\ g_2 \\ f_3 \\ g_3 \end{vmatrix}, \tag{7}$$

where (let $d/dr = R$)

$$A=\begin{vmatrix} ik & \frac{2m+1}{2r}+R & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1-2m}{2r}+R & -ik & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ik & \frac{2m+1}{2r}+R & 2k & \frac{i}{r} & \frac{1-2m}{r} & 0 \\ 0 & 0 & \frac{1-2m}{2r}+R & -ik & -\frac{i}{r} & 2k & 0 & -\frac{2m+1}{r} \\ 0 & 0 & -2k & -\frac{i}{r} & ik & \frac{2m+1}{2r}+R & -2iR & 0 \\ 0 & 0 & \frac{i}{r} & -2k & \frac{1-2m}{2r}+R & -ik & 0 & -2iR \\ 0 & 0 & \frac{2m-1}{r} & 0 & 2iR+\frac{2i}{r} & 0 & ik & \frac{2m+1}{2r}+R \\ 0 & 0 & 0 & \frac{2m+1}{r} & 0 & 2iR+\frac{2i}{r} & \frac{1-2m}{2r}+R & -ik \end{vmatrix}. \tag{8}$$

Let us transform this system to the cyclic basis [7–9]. Starting with the relations

$$\Psi_{\text{cycl}} = U\Psi_{\text{cyl}}, \quad \bar{\Psi}_{kB} = U_{BA}\Psi_{kA} = \Psi_{kA}\tilde{U}_{AB}, \quad \bar{\Psi} = \Psi\tilde{U}, \quad \Psi = \bar{\Psi}\tilde{U}^{-1}, \tag{9}$$

in 16-dimensional form $\bar{\Psi} = T_{16\times16}\Psi$ we have the needed repesentation; we use its truncated 8-form

$$T=\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 0 & i/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\sqrt{2} & 0 & i/\sqrt{2} & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/\sqrt{2} & 0 & i/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 & i/\sqrt{2} & 0 & 0 \end{vmatrix}. \tag{10}$$

Transition of the system to the cyclic basis is done by the rule

$$\bar{\Psi} = T\Psi, \quad (TAT^{-1})\bar{\Psi} = 2\sigma\bar{\Psi}, \quad \bar{A} = TAT^{-1}; \tag{11}$$

for the matrix $\bar{A}$ we obtain an explicit expression: then we get new equations

$$ikf_0 + \left(\frac{d}{dr}+\frac{m+1/2}{r}\right)g_0 = 2\sigma f_0, \quad \left(\frac{d}{dr}-\frac{m-1/2}{r}\right)f_0 - ikg_0 = 2\sigma g_0,$$

$$3ikf_1 + \sqrt{2}\left(\frac{d}{dr}+\frac{m-1/2}{r}\right)f_2 + \left(\frac{d}{dr}+\frac{m-1/2}{r}\right)g_1 = 2\sigma f_1,$$

$$\left(\frac{d}{dr}-\frac{m-3/2}{r}\right)f_1 + ikg_1 + \sqrt{2}\left(\frac{d}{dr}+\frac{m+1/2}{r}\right)g_2 = 2\sigma g_1,$$

$$\sqrt{2}\left(\frac{d}{dr}-\frac{m-3/2}{r}\right)f_1 + ikf_2 + \sqrt{2}\left(\frac{d}{dr}+\frac{m+1/2}{r}\right)f_3 + \left(\frac{d}{dr}+\frac{m+1/2}{r}\right)g_2 = 2\sigma f_2, \tag{12}$$

$$\left(\frac{d}{dr}-\frac{m-1/2}{r}\right)f_2+\sqrt{2}\left(\frac{d}{dr}-\frac{m-1/2}{r}\right)g_1-ikg_2+\sqrt{2}\left(\frac{d}{dr}+\frac{m+3/2}{r}\right)g_3=2\sigma g_2,$$
$$\sqrt{2}\left(\frac{d}{dr}-\frac{m-1/2}{r}\right)f_2-ikf_3+\left(\frac{d}{dr}+\frac{m+3/2}{r}\right)g_3=2\sigma f_3,$$
$$\left(\frac{d}{dr}-\frac{m+1/2}{r}\right)f_3+\sqrt{2}\left(\frac{d}{dr}-\frac{m+1/2}{r}\right)g_2-3ikg_3=2\sigma g_3.$$

With the use of shortening notations

$$a_{m-1/2}=\frac{d}{dr}+\frac{m-1/2}{r},\quad a_{m+1/2}=\frac{d}{dr}+\frac{m+1/2}{r},\quad a_{m+3/2}=\frac{d}{dr}+\frac{m+3/2}{r},$$
$$b_{m-1/2}=\frac{d}{dr}-\frac{m-1/2}{r},\quad b_{m+1/2}=\frac{d}{dr}-\frac{m+1/2}{r},\quad b_{m-3/2}=\frac{d}{dr}-\frac{m-3/2}{r},$$

we reduce the last system to the form

$$\begin{aligned}
&a_{m+1/2}g_0=(2\sigma-ik)f_0,\quad b_{m-1/2}f_0=(2\sigma+ik)g_0,\\
&\sqrt{2}a_{m-1/2}f_2+a_{m-1/2}g_1=(2\sigma-3ik)f_1,\\
&b_{m-3/2}f_1+\sqrt{2}a_{m+1/2}g_2=(2\sigma-ik)g_1,\\
&\sqrt{2}b_{m-3/2}f_1+\sqrt{2}a_{m+1/2}f_3+a_{m+1/2}g_2=(2\sigma-ik)f_2,\\
&b_{m-1/2}f_2+\sqrt{2}b_{m-1/2}g_1+\sqrt{2}a_{m+3/2}g_3=(2\sigma+ik)g_2,\\
&\sqrt{2}b_{m-1/2}f_2+a_{m+3/2}g_3=(2\sigma+ik)f_3,\\
&b_{m+1/2}f_3+\sqrt{2}b_{m+1/2}g_2=(2\sigma+3ik)g_3.
\end{aligned}\tag{13}$$

Method of projective operators. In order to solve the system (13) we will apply the method of projective operators [10]. To this end, let us start with the matrix (in the cyclic basis) $Y=\overline{J}^{12}=\sigma^{12}\otimes I+I\otimes\overline{j}^{12}$, whence it follows

$$Y\Psi_{\text{cycl}}=\begin{vmatrix}-\frac{i}{2}f_0 & -\frac{3i}{2}f_1 & -\frac{i}{2}f_2 & \frac{i}{2}f_3\\ \frac{i}{2}g_0 & -\frac{i}{2}g_1 & \frac{i}{2}g_2 & \frac{3i}{2}g_3\\ -\frac{i}{2}h_0 & -\frac{3i}{2}h_1 & -\frac{i}{2}h_2 & \frac{i}{2}h_3\\ \frac{i}{2}d_0 & -\frac{i}{2}d_1 & \frac{i}{2}d_2 & \frac{3i}{2}d_3\end{vmatrix}.\tag{14}$$

It is more convenient to have a 16-dimensional form of generator Y

$$\Psi=\Psi_{A(n)}=\begin{vmatrix}\Psi_{A(0)}\\ \Psi_{A(1)}\\ \Psi_{A(2)}\\ \Psi_{A(3)}\end{vmatrix},\quad \Psi_{A(0)}=\begin{vmatrix}f_0\\ g_0\\ h_0\\ d_0\end{vmatrix},\quad \Psi_{A(1)}=\begin{vmatrix}f_1\\ g_1\\ h_1\\ d_1\end{vmatrix},\quad \Psi_{A(2)}=\begin{vmatrix}f_2\\ g_2\\ h_2\\ d_3\end{vmatrix},\quad \Psi_{A(3)}=\begin{vmatrix}f_3\\ g_3\\ h_3\\ d_3\end{vmatrix};\tag{15}$$

for the relevant matrix Y we get the explicite expression. In which we can separate two similar 8-dimensional structures with respect to the variables:

$$f_0, g_0, \quad f_1, g_1, \quad f_2, g_2, \quad f_3, g_3, \qquad h_0, d_0, \quad h_1, d_1, \quad h_2, d_2, \quad h_3, d_3; \tag{16}$$

so the truncated matrix Y is defined by

$$Y\Psi = \begin{vmatrix} -i/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3i/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3i/2 \end{vmatrix} \begin{vmatrix} f_0 \\ h_0 \\ f_1 \\ h_1 \\ f_2 \\ h_2 \\ f_3 \\ h_3 \end{vmatrix}. \tag{17}$$

We verify that the minimal equation is valid [10]

$$(Y^2 + 1/4)(Y^2 + 9/4) = (Y + i/2)(Y - i/2)(Y + 3i/2)(Y - 3i/2) = 0, \tag{18}$$

which permits us to introduce four projective operators

$$P_1 = +\frac{i}{2}\left(Y - \frac{i}{2}\right)\left(Y^2 + \frac{9}{4}\right), \quad P_2 = -\frac{i}{2}\left(Y + \frac{i}{2}\right)\left(Y^2 + \frac{9}{4}\right),$$

$$P_3 = +\frac{i}{6}\left(Y^2 + \frac{1}{4}\right)\left(Y + \frac{3i}{2}\right), \quad P_4 = -\frac{i}{6}\left(Y^2 + \frac{1}{4}\right)\left(Y - \frac{3i}{2}\right),$$

with the properties $P_1 + P_2 + P_3 + P_4 = I$, $P_i^2 = P_i$; correspondingly, the wave function can be decomposed into the sum of 4 parts, $\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$. Their explicit form is readily found. Correspondingly, the truncated projective constituents are determined by the relations

$$\Psi_1 = \begin{vmatrix} f_0(r) \\ 0 \\ 0 \\ g_1(r) \\ f_2(r) \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \Psi_2 = \begin{vmatrix} 0 \\ g_0(r) \\ 0 \\ 0 \\ 0 \\ g_2(r) \\ f_3(r) \\ 0 \end{vmatrix}, \quad \Psi_3 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ g_3(r) \end{vmatrix}, \quad \Psi_4 = \begin{vmatrix} 0 \\ 0 \\ f_1(r) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}. \tag{19}$$

We divide 8 equations into 4 groups:

$$P_1, \quad a_{m+1/2} g_0 = (2\sigma - ik) f_0, \quad b_{m-3/2} f_1 + \sqrt{2} a_{m+1/2} g_2 = (2\sigma - ik) g_1,$$

$$\sqrt{2} b_{m-3/2} f_1 + \sqrt{2} a_{m+1/2} f_3 + a_{m+1/2} g_2 = (2\sigma - ik) f_2;$$

$$P_2, \quad b_{m-1/2} f_0 = (2\sigma + ik) g_0, \quad b_{m-1/2} f_2 + \sqrt{2} b_{m-1/2} g_1 + \sqrt{2} a_{m+3/2} g_3 = (2\sigma + ik) g_2,$$

$$\sqrt{2} b_{m-1/2} f_2 + a_{m+3/2} g_3 = (2\sigma + ik) f_3;$$

$$P_3, \quad b_{m+1/2} f_3 + \sqrt{2} b_{m+1/2} g_2 = (2\sigma + 3ik) g_3; \qquad P_4, \quad \sqrt{2} a_{m-1/2} f_2 + a_{m-1/2} g_1 = (2\sigma - 3ik) f_1.$$

According to the method by Fedorov – Gronskiy [10], each projective constituent is determined only by one function of the variable r:

$$\Psi_1=\begin{vmatrix} f_0\\0\\0\\g_1\\f_2\\0\\0\\0\end{vmatrix}\varphi_1(r),\quad \Psi_2=\begin{vmatrix} 0\\g_0\\0\\0\\0\\g_2\\f_3\\0\end{vmatrix}\varphi_2(r),\quad \Psi_3=\begin{vmatrix} 0\\0\\0\\0\\0\\0\\0\\g_3\end{vmatrix}\varphi_3(r),\quad \Psi_4=\begin{vmatrix} 0\\0\\f_1\\0\\0\\0\\0\\0\end{vmatrix}\varphi_4(r); \tag{20}$$

inside the columns some numerical coefficients stay. Besides, near each equation we have to impose additional differential constraints which permit us to transform the equations to the algebraic form:

$$P_1,\quad a_{m+1/2}\varphi_2 g_0=(2\sigma-ik)\varphi_1 f_0 \;\Rightarrow\; a_{m+1/2}\varphi_2=c_1\varphi_1,$$

$$b_{m-3/2}\varphi_4 f_1+\sqrt{2}a_{m+1/2}\varphi_2 g_2=(2\sigma-ik)\varphi_1 g_1 \;\Rightarrow\; b_{m-3/2}\varphi_4=c_2\varphi_1,\quad a_{m+1/2}\varphi_2=c_1\varphi_1,$$

$$\sqrt{2}b_{m-3/2}\varphi_4 f_1+\sqrt{2}a_{m+1/2}\varphi_2 f_3+a_{m+1/2}\varphi_2 g_2=(2\sigma-ik)\varphi_1 f_2 \;\Rightarrow\; b_{m-3/2}\varphi_4=c_2\varphi_1,\quad a_{m+1/2}\varphi_2=c_1\varphi_1;$$

$$P_2,\quad b_{m-1/2}\varphi_1 f_0=(2\sigma+ik)\varphi_2 g_0 \;\Rightarrow\; b_{m-1/2}\varphi_1=c_3\varphi_2,$$

$$b_{m-1/2}\varphi_1 f_2+\sqrt{2}b_{m-1/2}\varphi_1 g_1+\sqrt{2}a_{m+3/2}\varphi_3 g_3=(2\sigma+ik)\varphi_2 g_2 \;\Rightarrow\; b_{m-1/2}\varphi_1=c_3\varphi_2,\quad a_{m+3/2}\varphi_3=c_4\varphi_2,$$

$$\sqrt{2}b_{m-1/2}\varphi_1 f_2+a_{m+3/2}\varphi_3 g_3=(2\sigma+ik)\varphi_2 f_3 \;\Rightarrow\; b_{m-1/2}\varphi_1=c_3\varphi_2,\quad a_{m+3/2}\varphi_3=c_4\varphi_2;$$

$$P_3,\quad b_{m+1/2}\varphi_2 f_3+\sqrt{2}b_{m+1/2}\varphi_2 g_2=(2\sigma+3ik)\varphi_3 g_3 \;\Rightarrow\; b_{m+1/2}\varphi_2=c_5\varphi_3;$$

$$P_4,\quad \sqrt{2}a_{m-1/2}\varphi_1 f_2+a_{m-1/2}\varphi_1 g_1=(2\sigma-3ik)\varphi_4 f_1 \;\Rightarrow\; a_{m-1/2}\varphi_1=c_6\varphi_4.$$

In this way, from the above we arrive at the following algebraic system

$$\begin{aligned}
&c_1 g_0=(2\sigma-ik)f_0,\quad c_2 f_1+\sqrt{2}c_1 g_2=(2\sigma-ik)g_1,\\
&\sqrt{2}c_2 f_1+\sqrt{2}c_1 f_3+c_1 g_2=(2\sigma-ik)f_2,\quad c_3 f_0=(2\sigma+ik)g_0,\\
&c_3 f_2+\sqrt{2}c_3 g_1+\sqrt{2}c_4 g_3=(2\sigma+ik)g_2,\quad \sqrt{2}c_3 f_2+c_4 g_3=(2\sigma+ik)f_3,\\
&c_5 f_3+\sqrt{2}c_5 g_2=(2\sigma+3ik)g_3,\quad \sqrt{2}c_6 f_2+c_6 g_1=(2\sigma-3ik)f_1,
\end{aligned} \tag{21}$$

and at the first-order constraints

$$\begin{aligned}
&a_{m+1/2}\varphi_2=c_1\varphi_1,\quad b_{m-1/2}\varphi_1=c_3\varphi_2,\quad \text{let}\quad c_3=c_1;\\
&a_{m-1/2}\varphi_1=c_6\varphi_4,\quad b_{m-3/2}\varphi_4=c_2\varphi_1,\quad \text{let}\quad c_6=c_2;\\
&a_{m+3/2}\varphi_3=c_4\varphi_2,\quad b_{m+1/2}\varphi_2=c_5\varphi_3,\quad \text{let}\quad c_5=c_4.
\end{aligned} \tag{22}$$

Whence we derive the 2-nd-order equations for separate functions

$$\begin{aligned}
&\left(a_{m+1/2}b_{m-1/2}-c_1^2\right)\varphi_1=0,\quad \left(b_{m-3/2}a_{m-1/2}-c_2^2\right)\varphi_1=0,\\
&\left(b_{m-1/2}a_{m+1/2}-c_1^2\right)\varphi_2=0,\quad \left(a_{m+3/2}b_{m+1/2}-c_4^2\right)\varphi_2=0,\\
&\left(b_{m+1/2}a_{m+3/2}-c_4^2\right)\varphi_3=0,\quad \left(a_{m-1/2}b_{m-3/2}-c_2^2\right)\varphi_4=0.
\end{aligned} \tag{23}$$

Explicitly they read

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(1-2m)^2}{4r^2}-c_1^2\right)\varphi_1=0,\quad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(1-2m)^2}{4r^2}-c_2^2\right)\varphi_1=0 \;\Rightarrow\; c_1^2=c_2^2,$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(1+2m)^2}{4r^2}-c_1^2\right)\varphi_2=0,\quad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(1+2m)^2}{4r^2}-c_4^2\right)\varphi_2=0 \;\Rightarrow\; c_1^2=c_4^2,$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(3+2m)^2}{4r^2}-c_4^2\right)\varphi_3=0,\qquad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(3-2m)^2}{4r^2}-c_2^2\right)\varphi_4=0.$$

Therefore, there exists only one independent parameter $c_2^2=c_4^2=c_1^2$; and the above equations take the form

$$\begin{gathered}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(m-1/2)^2}{r^2}-c_1^2\right)\varphi_1=0,\qquad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(m+1/2)^2}{r^2}-c_1^2\right)\varphi_2=0,\\ \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(m+3/2)^2}{r^2}-c_1^2\right)\varphi_3=0,\qquad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(m-3/2)^2}{r^2}-c_1^2\right)\varphi_4=0.\end{gathered}\tag{24}$$

In the variable $x=ic_1r$, they turn to a Bessel form

$$\begin{gathered}\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1-\frac{(m-1/2)^2}{x^2}\right)\varphi_1=0,\qquad \varphi_1=J_{\pm(m-1/2)}(x);\\ \left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1-\frac{(m+1/2)^2}{x^2}\right)\varphi_2=0,\qquad \varphi_2=J_{\pm}(m+1/2)(x);\\ \left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1-\frac{(m+3/2)^2}{x^2}\right)\varphi_3=0,\qquad \varphi_3=J_{\pm(m+3/2)}(x);\\ \left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1-\frac{(m-3/2)^2}{x^2}\right)\varphi_4=0,\qquad \varphi_4=J_{\pm(m-3/2)}(x).\end{gathered}\tag{25}$$

This analysis can be readily extended to the case of the presence of the external magnetic field. In fact, we should make one formal change field $m\Rightarrow m+eBr^2/2$ (let $eB\Rightarrow B$); so we get operators

$$\begin{gathered}a_{\mu-1/2}=\frac{d}{dr}+\frac{m-1/2+Br^2/2}{r},\qquad a_{\mu+1/2}=\frac{d}{dr}+\frac{m+1/2+Br^2/2}{r},\\ a_{\mu+3/2}=\frac{d}{dr}+\frac{m+3/2+Br^2/2}{r},\qquad b_{\mu-1/2}=\frac{d}{dr}-\frac{m-1/2+Br^2/2}{r},\\ b_{\mu+1/2}=\frac{d}{dr}-\frac{m+1/2+Br^2/2}{r},\qquad b_{\mu-3/2}=\frac{d}{dr}-\frac{m-3/2+Br^2/2}{r}.\end{gathered}\tag{26}$$

Correspondingly, the equations (25) become more complicated; at this there arise the constraints

$$c_2^2=c_1^2+2B,\quad c_4^2=c_1^2-2B,\tag{27}$$

therefore we have four 2-nd-order equations (let $c_1^2=X$):

$$\begin{gathered}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{4}B^2r^2-\frac{(m-1/2)^2}{r^2}-B(m+1/2)-X\right)\varphi_1=0,\\ \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{4}B^2r^2-\frac{(m+1/2)^2}{r^2}-B(m-1/2)-X\right)\varphi_2=0,\\ \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{4}B^2r^2-\frac{(m+3/2)^2}{r^2}-B(m-3/2)-X\right)\varphi_3=0,\\ \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{4}B^2r^2-\frac{(m-3/2)^2}{r^2}-B(m+3/2)-X\right)\varphi_4=0.\end{gathered}\tag{28}$$

They have one and the same structure. Let us consider in detail the last equation. In the variable $r^2 = x$, we get

$$\left[\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{B^2}{16}-\frac{(m-3/2)^2}{4x^2}-\frac{B(m+3/2)+X}{4x}\right]\varphi_4=0,$$

which belongs to a hypergeometric type. In the vicinity of $x = 0$ and $x \to \infty$ their solutions behave

$$\varphi_4 = x^A, \quad A=\pm|m-3/2|/2; \quad \varphi_4=e^{Dx}, \quad D=\pm B/4;$$

for the bound states we are to use positive values of A and (assuming that $B > 0$) a negative value of D. General solutions are searched in the form $\varphi_4 = x^A e^{Dx} F_4(x)$; further we derive

$$\left\{x\frac{d^2}{dx^2}+[(2A+1)+2Dx]\frac{d}{dx}-\frac{B(m+3/2)+X-4D(2A+1)}{4}\right\}F_4=0.$$

In the variable $2Dx = -z$, we obtain equation

$$z\frac{d^2}{dz^2}F_4+(|m-3/2|+1-z)\frac{d}{dz}F_4-\left(\frac{(m+3/2)+|m-3/2|+1}{2}+\frac{X}{2B}\right)F_4=0 \tag{29}$$

of hypergeometric type. The polynomial condition $\alpha = -n_4$ leads to

$$\frac{(m+3/2)+|m-3/2|+1}{2}+n_4+\frac{X}{2B}=0, \quad n_4=0,1,2,...; \tag{30}$$

with notation

$$N_4=\frac{(m+3/2)+|m-3/2|+1}{2}+n_4,$$

the quantization rule reads $X = -2BN_4 < 0$. Three remaining equations are studied similarly. Thus we get the following results

$$\begin{aligned}
&F_1, \quad N_1=\frac{(m+1/2)+|m-1/2|+1}{2}+n_1, \quad X=-2BN_1<0;\\
&F_2, \quad N_2=\frac{(m-1/2)+|m+1/2|+1}{2}+n_2, \quad X=-2BN_2<0;\\
&F_3, \quad N_3=\frac{(m-3/2)+|m+3/2|+1}{2}+n_3, \quad X=-2BN_3<0;\\
&F_4, \quad N_4=\frac{(m+3/2)+|m-3/2|+1}{2}+n_4, \quad X=-2BN_4<0.
\end{aligned} \tag{31}$$

Solving the algebraic system for the case of a free particle. For a free particle, we have one parameter, $c_1,...,c_6 = C$; and the matrix form of the system is $A\Phi = 0$,

$$A=\left|\begin{array}{cccccccc}
-(2\sigma-ik) & C & 0 & 0 & 0 & 0 & 0 & 0\\
C & -(2\sigma+ik) & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -(2\sigma-3ik) & C & \sqrt{2}C & 0 & 0 & 0\\
0 & 0 & C & -(2\sigma-ik) & 0 & \sqrt{2}C & 0 & 0\\
0 & 0 & \sqrt{2}C & 0 & -(2\sigma-ik) & C & \sqrt{2}C & 0\\
0 & 0 & 0 & \sqrt{2}C & C & -(2\sigma+ik) & 0 & \sqrt{2}C\\
0 & 0 & 0 & 0 & \sqrt{2}C & 0 & -(2\sigma+ik) & C\\
0 & 0 & 0 & 0 & 0 & \sqrt{2}C & C & -(2\sigma+3ik)
\end{array}\right|.$$

The equation $\det A = 0$ leads to $(9C^2 - 9k^2 - 4\sigma^2)(C^2 - k^2 - 4\sigma^2)^3 = 0$. The roots are as follows

$$\sigma_1 = -\frac{3}{2}\sqrt{C^2 - k^2}, \quad \sigma_2 = +\frac{3}{2}\sqrt{C^2 - k^2}, \quad \sigma_3 = -\frac{1}{2}\sqrt{C^2 - k^2}, \quad \sigma_4 = \frac{1}{2}\sqrt{C^2 - k^2}. \qquad (32)$$

Solutions of the algebraic system at these four values can be readily found.

Solving the algebraic system in the presence of the magnetic field. In the presence of the magnetic field, taking in mind the identities $c_3 = c_1$, $c_6 = c_2 = \sqrt{X + 2B}$, $c_5 = c_4 = \sqrt{X - 2B}$, $c_1 = \sqrt{X}$, in dimensionless quantities $\sigma / \sqrt{X} = \Sigma$, $k / \sqrt{X} = K$, $B / X = b < 0$, $X = -2BN < 0$, we get

$$\begin{aligned}
&g_0 = (2\Sigma - iK) f_0, \quad f_0 = (2\Sigma + iK) g_0, \\
&\sqrt{2}\sqrt{1+2b} f_2 + \sqrt{1+2b} g_1 = (2\Sigma - 3iK) f_1, \\
&\sqrt{1+2b} f_1 + \sqrt{2} g_2 = (2\Sigma - iK) g_1, \\
&\sqrt{2}\sqrt{1+2b} f_1 + \sqrt{2} f_3 + g_2 = (2\Sigma - iK) f_2, \\
&f_2 + \sqrt{2} g_1 + \sqrt{2}\sqrt{1-2b} g_3 = (2\Sigma + iK) g_2, \\
&\sqrt{2} f_2 + \sqrt{1-2b} g_3 = (2\Sigma + iK) f_3, \\
&\sqrt{1-2b} f_3 + \sqrt{2}\sqrt{1-2b} g_2 = (2\Sigma + 3iK) g_3.
\end{aligned}$$

It can be presented in the matrix form $A\Phi = 0$; from the equation $\det A = 0$ we obtain

$$(K^2 + 4\Sigma^2 - 1)^2 \Big(-36b^2 - 96ibK\Sigma + (K^2 + 4\Sigma^2 - 1)(9K^2 + 4\Sigma^2 - 9)\Big) = 0; \qquad (33)$$

this equation is factorized. The roots for the simple equation are

$$\Sigma_1 = +\frac{1}{2}\sqrt{1 - K^2}, \quad \Sigma_2 = -\frac{1}{2}\sqrt{1 - K^2}, \quad \text{multiplicity 2.}$$

Let us transform the equation $K^2 + 4\Sigma^2 - 1 = 0$ to initial parameters:

$$\Sigma = \frac{\sigma}{i\sqrt{2BN}}, \quad K = \frac{k}{i\sqrt{2BN}},$$

then we obtain $2BN + k^2 + 4\sigma^2 = 0$. In the variable $Z = i\sigma$, this equation reads $2BN + k^2 - 4Z^2 = 0$. The numerical study at two sets of parameters gives

	−0.866025,	0.866025,		−2.29129,	2.29129,
	−1.11803,	1.11803,		−3.20156,	3.20156,
	−1.32288,	1.32288,		−3.90512,	3.90512,
	−1.5,	1.5,		−4.5,	4.5,
	−1.65831,	1.65831,		−5.02494,	5.02494,
$B = 1,\ k = 1,\ N = 1,...,10,$	−1.80278,	1.80278,	$B = 10,\ k = 1,\ N = 1,...,10,$	−5.5,	5.5,
	−1.93649,	1.93649,		−5.93717,	5.93717,
	−2.06155,	2.06155,		−6.34429,	6.34429,
	−2.17945,	2.17945,		−6.72681,	6.72681,
	−2.29129,	2.29129;		−7.08872,	7.08872.

The second equation is

$$16\Sigma^4 + 40(K^2 - 1)\Sigma^2 - 96ibK\Sigma + 9\Big((K^2 - 1)^2 - 4b^2\Big) = 0;$$

its solutions are readily found in the analytical form. However, the numerical study will be more convenient. To this end let us turn back to initial variables; which give (it is convenient to use the new variable $Z = i\sigma$):

$$16Z^4 - 40(2BN + k^2)Z^2 - 96BkZ + 9\Big(2B(N-1) + k^2\Big)\Big(2B(N+1) + k^2\Big) = 0. \tag{34}$$

The numerical example study gives

$$B = 1, \quad k = 1, \quad N = 1,...,10, \quad \begin{array}{llll} -1.86147, & -1.5, & 0.332551, & 3.02892, \\ -3.04142, & -1.37523, & 0.775656, & 3.641, \\ -3.75612, & -1.5, & 1.07434, & 4.18178, \\ -4.33759, & -1.63742, & 1.30616, & 4.66885, \\ -4.84306, & -1.77134, & 1.5, & 5.1144, \\ -5.29713, & -1.89911, & 1.66928, & 5.52696, \\ -5.71326, & -2.0206, & 1.82126, & 5.9126, \\ -6.09982, & -2.13628, & 1.9603, & 6.27581, \\ -6.46245, & -2.24675, & 2.08921, & 6.61998, \\ -6.80517, & -2.35254, & 2.20996, & 6.94775. \end{array}$$

Conclusions. The eigenvalue problem for the generalized helicity operator for a spin 3/2 particle in the presence of the uniform magnetic field has been solved. After separating the variables in the basis of cylindrical coordinates, the system of 16 differential equations in the variable *r* is derived. This system is solved with the use of the method of projective operators. All 16 variables were expressed in terms of only 4 distinguished functions, which are constructed in terms of confluent hypergeometric functions. Further the problem reduces to studying the linear algebraic system for 16 algebraic variables. In the end, we derive equations of the second and the fourth order, their roots determine the possible eigenvalues of the helicity operator. The developed method can be extended to the field with spin 2.

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