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STRUCTURAL APPROACH TO THE ADAPTIVE OPTICS TOMOGRAPHY PROBLEM: MULTI-CONJUGATE TURBULENCE CORRECTION

Abstract. The necessity of correcting the perturbations related to light propagation in the turbulent atmosphere and obtaining the diffraction-limited images requires computer processing of a large information flow in real time. Here the straightforward approaches, which ignore the structural properties of the atmospheric turbulence and the active optical system itself, lead to algorithms with complexity proportional to the fourth power of the aperture diameter. In the paper it is shown that taking into account the structural properties of atmospheric turbulence as well as the active optical system itself allows to significantly reduce the complexity coming close to the second power of the diameter. The approach is proposed, which is based on the efficient construction of deformable mirror commands in a multi-conjugate adaptive system in the case of already known turbulence volume distribution taking into account structural properties of the active optical system itself. The efficiency of the proposed approach is demonstrated by the example of the system of adaptive optics of a 30-meter telescope pupil with a 6-layer atmospheric turbulence model.

Keywords: Multi-Conjugate Adaptive Optics, tomographic phase reconstruction, structured matrices, real-time signal processing

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СТРУКТУРНЫЙ ПОДХОД К РЕШЕНИЮ ТОМОГРАФИЧЕСКОЙ ЗАДАЧИ АДАПТИВНОЙ ОПТИКИ: МНОГОСОПРЯЖЕННАЯ КОРРЕКЦИЯ ТУРБУЛЕНТНОСТИ

Аннотация. Необходимость корректировки искажений, связанных с распространением света в турбулентной атмосфере, и получения изображения дифракционного качества требует компьютерной обработки больших потоков информации в реальном времени. При этом прямолинейные подходы, игнорирующие структурные свойства атмосферной турбулентности и самой активной оптической системы, приводят к алгоритмам со сложностью, пропорциональной четвертой степени диаметра апертуры. Показано, что учет структурных свойств атмосферной турбулентности, а также самой активной оптической системы позволяют существенно снизить вычислительную сложность, приблизившись по порядку к квадрату диаметра апертуры. Предложен подход, основанный на эффективном построении команд для деформируемых зеркал многосопряженной адаптивной системы при уже известном объемном распределении турбулентности с учетом структурных свойств самой активной оптической системы. Эффективность предложенного подхода продемонстрирована на примере системы адаптивной оптики 30-метрового телескопа с шестислойной моделью атмосферной турбулентности.

Ключевые слова: многосопряженная адаптивная оптика, томографическое фазовое восстановление, структурированные матрицы, обработка сигналов в реальном времени

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Introduction. Adaptive Optics as a relatively new trend in the modern optical systems allows to correct the perturbations related to light propagation in the turbulent atmosphere obtaining the diffraction-limited images. Because it requires computer processing of a large information flow in real time, a key problem lays in finding the mathematical algorithms with reduced computational complexity.

In this work we extend our previous paper [1] on the *structured control* of the Multi-Conjugate Adaptive Optics system, which allows to dramatically reduce the computational complexity. We remind that the Adaptive Optics as a method of real-time correction of dynamic atmospheric turbulence effects on large-scale optical systems like astronomical telescopes suffers from the *curse of complexity* [2]: *computational complexity grows fast with increasing the controlled system scale.*

We had shown that computational complexity of the problem of tomographic turbulence volume reconstruction is $\mathcal{O}(D^4)$ in case when no structural properties are taken into account but can be reduced to $\mathcal{O}(D^2)$ when the structural properties are employed. This result was obtained for the *first stage* of the Multi-Conjugate Adaptive Optics *tomographic problem,* namely, the *Phase Estimation.* Now we proceed with the AO system *structural analysis* for the *second stage* of the tomographic problem, *Phase Fitting.* What we will show is that the Phase Fitting, after certain auxiliary operations, can be reduced to a form with the *Sparse Toeplitz* structure, and thus has $\mathcal{O}(D^2)$-complexity.

We remind a reader the Multi-Conjugate Adaptive Optics (MCAO) system design (Fig. 1) and the related computational problem of the *phase reconstruction*:

$$\mathbf{a} = \mathbb{R}\mathbf{s}, \tag{1}$$

where **s** is the vector of Wave Front Sensor (WFS) measurements, **a** is the vector of Deformable Mirror (DM) correction commands; R is the reconstructor matrix. The reconstructor matrix is, in turn, split into the product

$$\mathbb{R} = \mathbb{F}\mathbb{E},$$

$$\mathbb{F} = \left(\mathbb{H}_a^T \mathbb{W} \mathbb{H}_a + \alpha \mathbb{I}\right)^{-1} \mathbb{H}_a^T \mathbb{W} \mathbb{H}_\phi, \quad \mathbb{E} = \left\langle \boldsymbol{\phi} \mathbf{s}^T \right\rangle \left\langle \mathbf{s}\mathbf{s}^T \right\rangle^{-1}, \tag{2}$$

where $\mathbb{E}$ is the *estimation* matrix, the structure of which was described in detail in [1]. Now we concentrate our attention on the *fitting matrix* $\mathbb{F}$. The goal is to reduce computational complexity to its “natural” value by employing the internal structure of the *phase fitting* problem.

Structured Fitting Step solution. We describe first the way fitting matrix elements are computed. Phase in the entrance pupil plane is presented as a *finite linear combination*

$$\phi(\mathbf{r}) = \sum_{i=1}^{N} h_i h_i(\mathbf{r}) \overset{\Delta}{=} \mathbf{h}^T(\mathbf{r})\mathbf{h}, \quad \mathbf{r} = [xy]^T, \tag{3}$$

where Δ means “by definition”; $\mathbf{h}(\mathbf{r}) = h_p(\mathbf{r})|_{p=1}^{N\to\infty}$ is the *basis function set* defined in the entrance pupil *xy*-plane in either phase screen, DM or WFS plane.

Analogously, on each turbulence phase screen a phase linear approximation is defined

$$\phi(\mathbf{r}) = \sum_{i=1}^{N} \phi_i \phi_i(\mathbf{r}) \overset{\Delta}{=} \boldsymbol{\phi}^T(\mathbf{r})\boldsymbol{\phi}, \tag{4}$$

where $\boldsymbol{\phi}(\mathbf{r}) = \phi_i(\mathbf{r})|_{i=1}^{N\to\infty}$ is the basis defined on each screen. A deformable mirror optically conjugated to a certain altitude above the telescope is considered as another phase screen, which introduces a phase correction

$$\delta\phi(\mathbf{r}) = \sum_{i=1}^{N} a_i f_i(\mathbf{r}) \overset{\Delta}{=} \mathbf{f}^T(\mathbf{r})\mathbf{a}, \tag{5}$$

where $\mathbf{f}(\mathbf{r}) = f_i(\mathbf{r})|_{i=1}^{N}$ is the set of the DM *influence functions* that correspond to actions of the DM *actuators* (degrees of freedom).

Also defined is the *Projection Operator* $\mathscr{H}(\phi)$, which, in geometrical optics approximation, is a central projection of a phase screen plane down to the pupil plane as it is shown on Fig. 2, left. The projection operator is written as

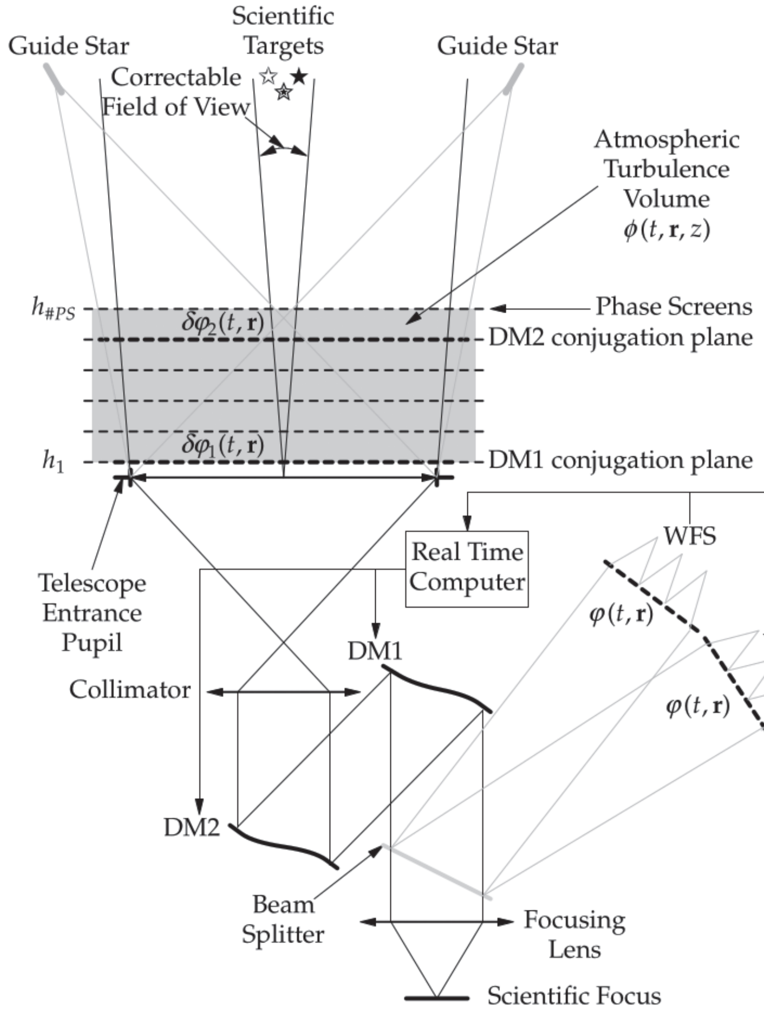


Fig. 1. Optical scheme of a Multi-Conjugate AO system with (at least) two *Deformable Mirrors*. Dynamic 3D *phase distortion* $\phi(t,\mathbf{r},z)$ is probed by passing through the *turbulence volume* of the light from multiple light sources, the *Guide Stars* (GS), natural or artificial. The *Wave Font Sensors* measure a set of 2D *wavefronts* $\varphi_g(t,\mathbf{r})|_{g=1}^{\#Gs}$ accumulated along the paths from multiple Guide Stars to the telescope pupil, which provides information allowing to *tomographically reconstruct* $\phi(t,\mathbf{r},z)$ in the volume of turbulent air above the telescope. Multiple *Deformable Mirrors* optically conjugated over several altitudes within the turbulence volume generate the *phase corrections* $\delta\phi_d(t,\mathbf{r})|_{d=1}^{\#DM}$ in an attempt to compensate the phase distortion not in a plane but in the entire volume, which helps to achieve the turbulence compensation over a broader FoV containing the *Scientific Targets* of interest. Phase distortion reconstruction from the WFS measurements and generation of DM correction commands are performed by the *Real Time Computer*. The light from Scientific Targets is drawn in black, from Guide Stars one is drawn in grey. Vector *r* stands for *x,y*-coordinates in either phase screen, DM or WFS plane

$$\mathscr{H}_{ij}(\phi):\phi(\mathbf{r}')\to\phi(\mathbf{r})=\phi(a_{ij}\mathbf{r}+\mathbf{b}_{ij}),\quad a_{ij}=1-h_j/z_i,\quad \mathbf{b}_{ij}=\frac{h_j}{z_i}[x_i,y_i],\tag{6}$$

where $\mathbf{r}'$ is defined on the phase screen, $\mathbf{r}$ on the pupil; $\mathbf{r}_i=(x_i,y_i,z_i)$ are the coordinates of the light source (the Scientific Target on Fig. 1).

The finite-dimensional approximation of the Projection Operator is a matrix acting on the basis weighting coefficients in (4) and getting the weighting coefficients in (3):

$$\mathscr{H}_{ij}\to\mathbb{H}_{ij}=\left(\mathbf{h}(\mathbf{r}),\mathbf{h}^T(\mathbf{r})\right)^{-1}\left(\mathbf{h}(\mathbf{r}),\boldsymbol{\phi}^T(a_{ij}\mathbf{r}+\mathbf{b}_{ij})\right),\quad \left(a(\mathbf{r}),b(\mathbf{r})\right)=\int_p a(\mathbf{r})b(\mathbf{r})d(\mathbf{r}).\tag{7}$$

Each matrix is a block of the H_ϕ- or H_a-matrix for ith Scientific Target and jth phase screen/DM. Consider the simplest case when the phase screen basis $\phi_s(\mathbf{r})|_{p=s}^{S}$ is bilinear splines, and the pupil basis

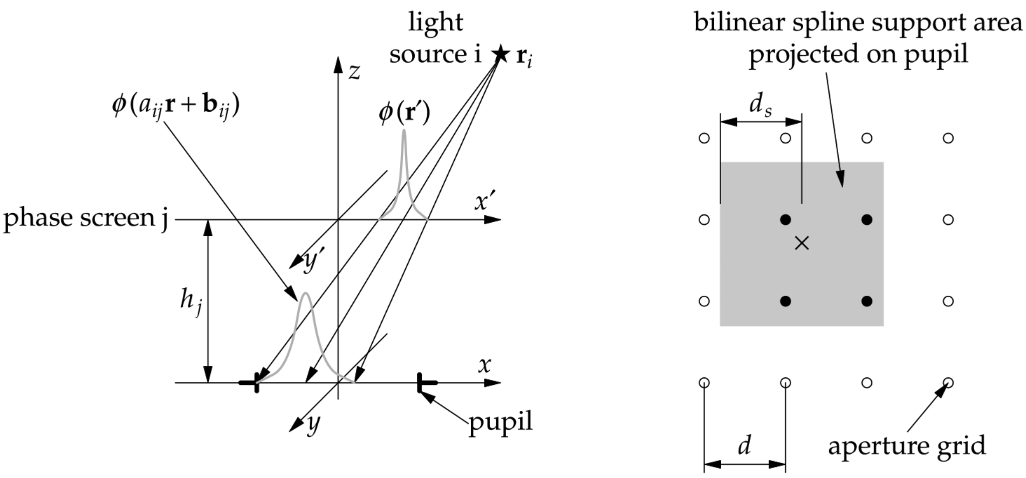


Fig. 2. Left: geometry of phase basis function projection along the light source with coordinates r onto the telescope entrance pupil. Right: geometrical pattern for H-matrix elements computation leading to Block - Toeplitz with Toeplitz Blocks (BTTB) matrix structure. Black points are the only ones influenced by a spline projection. If the spline width ds is equal to the pupil point grid spacing d, the corresponding H-matrix is BTTB

$\phi_p(\mathbf{r})|_{p=k}^{P}$ is just zero-order splines (constants within the support), i. e. the low-order shift-invariant basis functions. In this case the H-matrix elements are just values of projected phase screen basis functions on the phase grid points as it is shown on Fig. 2, right. Because of shift-invariance and on condition that $d = d_s$ the H-matrix becomes BTTB with its kernel size of 2 × 2, which makes the corresponding matrix-vector multiplication extremely fast.

Estimation-Fitting mismatch. We notice, however, that condition $d = d_s$ is generally not satisfied. Indeed, let's consider the H_ϕ-matrix first. It maps the weighting coefficients from the phase screens on which the phase was estimated from the Wavefront Sensor (WFS) Measurements during the Estimation step onto the pupil. The phase screen spacings were chosen to be equal $a_{ij}d$, where d is the WFS subaperture spacing and a_{ij} are the scaling factors computed from ith Guide Star, not the ith Scientific Target as it is necessary for the Fitting Step. Since the Guide stars and Scientific targets are generally different by altitude (the Laser Guide Stars are at finite altitude, Scientific Targets are usually at infinity), the phase screen grid spacings do not allow obtaining the BTTB-structure.

A simple way to fix this problem is to re-interpolate the estimated phase maps to new point grids with spacings needed for efficient fitting. This, unfortunately, results in an additional interpolation error. To minimize this error it is desirable to take the new spacing on the pupil and the corresponding spacings on the screens smaller than the WFS subaperture spacing. Note that re-interpolation is $\mathcal{O}(D^2)$-operation, which is done screen-wise, so its complexity is small.

An additional difficulty is that the DM influence functions in Eq. (5) are not necessarily shift-invariant. This may require an additional step. Instead of the natural basis made of the influence functions each DM phase screen can be supplied with a desired shift-invariant basis, which allows to obtain the BTTB-structure for the H_a-matrix in (2). To return from the desired basis to the natural influence functions, we introduce the *Influence Matrix* P: the matrix, columns of which contain the approximation weights in the desired basis for each DM influence function. Then the DM commands are found by solving the linear equation

$$\mathbb{P}\mathbf{c} = \mathbf{a}, \tag{8}$$

where **a** is taken from (1); **c** is the command vector to be applied directly to DMs. This adds an additional approximation error and additional complexity, which is not large because Eq. (8) is applied DM-wise, but noticeable because P is generally a full matrix. To avoid this step, it is possible to build deformable mirrors in a way their influence functions are shift-invariant by design. For instance, the Micro Electro Mechanical (MEMS) deformable mirrors by design have approximately shift-invariant influence functions.

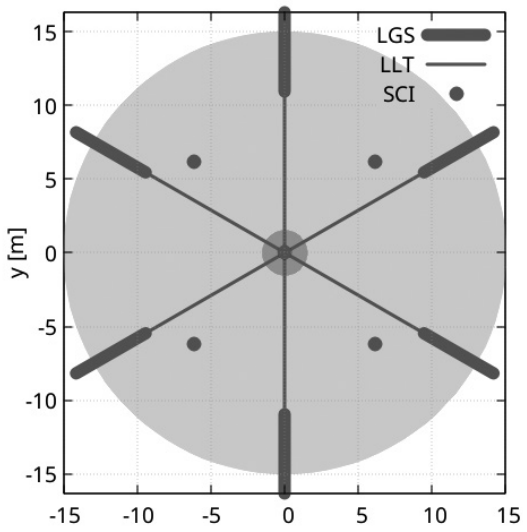


Fig. 3. 30-meter telescope pupil and the star constellation for the tested AO system. *LGS* is the Laser Guide Star, *SCI* is the Scientific Direction, *LLT* is the Laser Launch Telescope

Large scale AO fitting step simulation. In this section we demonstrate the complexity reduction effect by comparing solutions of the same phase fitting problem 7 either using a straightforward full-matrix approach or the *Sparse Toeplitz* one presented in this paper. Attention is focused on:

1) correctness of the computer code;

2) computational speedup,

ignoring the proceeding *Phase Estimation* step by assuming that the phase on a set of estimation phase screens is known in advance.

A 30-meter telescope AO system with the following parameters was taken as a *large-scale* testing case:

– a simple round 30-meter pupil;

– 6-layer Cerro Tololo atmospheric turbulence model [3];

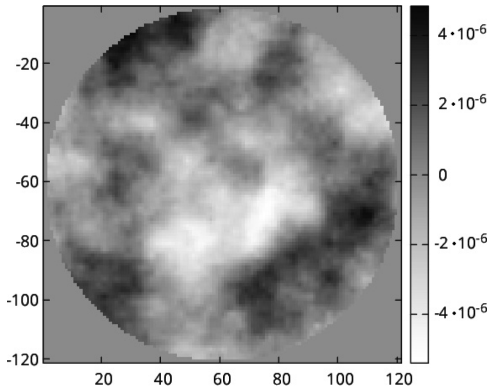
– 5 Scientific Natural Star Targets, one on-axis and four 20″ off-axis as it is shown on Fig. 3. The turbulence random phase is generated on each phase screen using the Fractal Method [4]. These phase maps are considered as a result of the phase estimation made on the preceding *Estimation Step.*

Only fitting to this phase by two Deformable Mirrors located at altitudes 0 and 5 km was simulated as a solution of the linear equation system:

$$\left(\mathbb{H}_a^T \mathbb{W} \mathbb{H}_a + \alpha \mathbb{I}\right)\mathbf{a} = \mathbb{H}_a^T \mathbb{W} \mathbb{H}_\phi \hat{\boldsymbol{\phi}}, \tag{9}$$

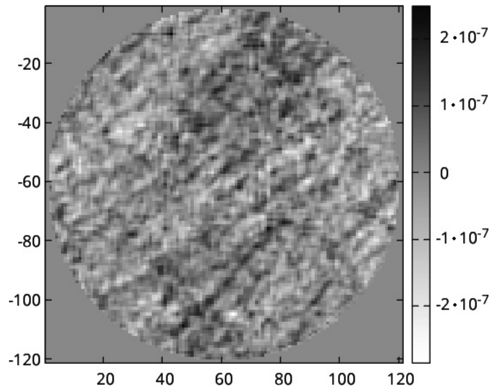
taken from (2). Here $\hat{\boldsymbol{\phi}}$ is the "estimated" phase vector generated on 6 phase screens. The phase screen spacings were also taken from the Estimation Step: they are equal to the spacing of the Wave Front Sensor subaperture grid corrected for the projection along the Guide Stars. Since this creates the Estimation / Fitting Mismatch described above, the phase maps where re-interpolated to the maps with spacings equal to the half subaperture grid spacing (a smaller spacing for accuracy) and then back-projected along the Scientific Targets. This additional reinterpolation step requires extra time that needs to be measured. The weights of the Scientific Targets were taken to be equal, so in Eq. (9) $\mathbb{W} = \mathbb{I}$. The solution of Eq. (9) is done iteratively using the standard Conjugate Gradients Method, which requires only matrix-vector multiplications that can be performed matrix-less by the Sparse Toeplitz convolution algorithm described in [1].

Figs. 4, 5 demonstrate the correctness of the *Phase Fitting* algorithm *Sparse Toeplitz* implementation. As to its efficiency, it can be compared with the classical *full-matrix* implementation of the same algorithm: Eq. (2) full matrix size for the given AO system is approximately 22000 × 66000. The computation time obtained on a single core, without parallelization, Intel i3 2500 MHz processor is:



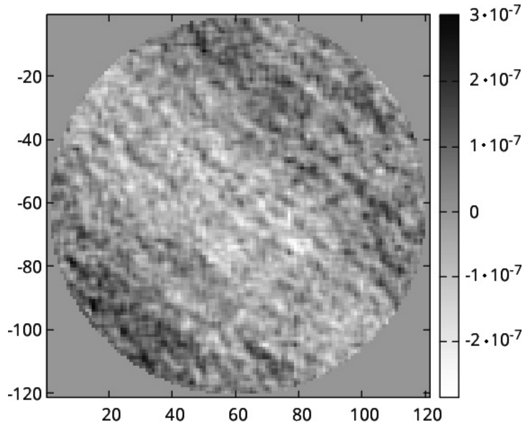
RMS ≈ *2000 nm*

a



RMS ≈ *57 nm*

b



RMS ≈ *75 nm*

c

Fig. 4. The *Piston,Tip,Tilt* -removed turbulent phase in the entrance pupil before and after phase fitting: *a* is the phase on-axis before correction, the total $C_n^2 = 7 \cdot 10^{-13}$ m$^{-2/3}$; *b* is the phase on-axis after correction; *c* is the phase 20″ off-axis after correction. The *Root Mean Square* error (RMS) is given in units of the *Optical Path Difference* (OPD)

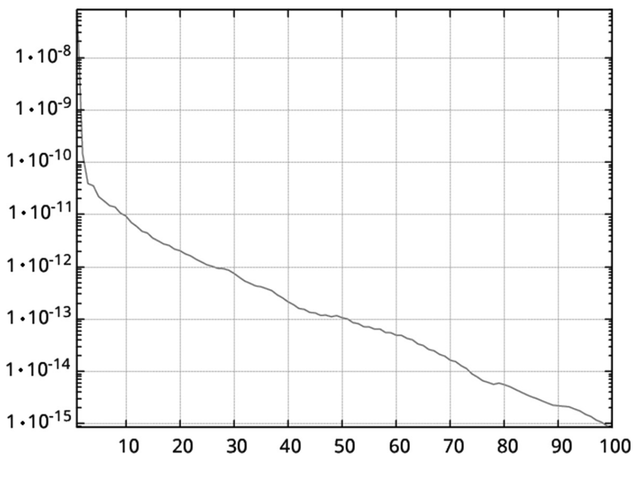


Fig. 5. *Conjugate Gradients* residual error evolution over iterations

– 100 iterations of Sparse Toeplitz Algorithm + Re-interpolation 0.052 s;

– Direct matrix-vector multiplication 1.2 s.

The simulation results show obvious advantage of the Sparse Toeplitz structural approach to the *Fitting Step* of the MCAO *Tomography* problem even taking into account that the Conjugate

Gradient algorithm convergence is rather slow, same as the convergence in the *Estimation Step* as it was shown in [1]. Finding the efficient preconditioner for both Estimation and Fitting Steps of the MCAO Tomography is a goal for the future work.

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