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SOLUTION OF A MIXED PROBLEM FOR THE WAVE EQUATION WITH A DYNAMIC BOUNDARY CONDITION AND DISCONTINUOUS INITIAL DATA IN THE FIRST QUADRANT OF THE PLANE

Abstract. We consider a mixed problem for the one-dimensional wave equation in the first quarter of the plane with discontinuous initial and boundary conditions arising from the theory of longitudinal impact on an elastic semi-infinite rod with elastic fastening at the left end. We propose a new approach to solve the problem, which constructs the solution as a limit of classical solutions. The proposed method does not use any additional conditions that were required in previous methods providing a more direct and rigorous construction. We find the solution in an explicit analytical form as a piecewise continuous function. For the problem under consideration, the uniqueness of the solution is proved and the necessary and sufficient matching conditions are established under which the solution exists.

Keywords: wave equation, mixed problem, discontinuous initial data, dynamic boundary condition, matching conditions, conjugation conditions, longitudinal impact, elastic rod

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РЕШЕНИЕ СМЕШАННОЙ ЗАДАЧИ ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ С ДИНАМИЧЕСКИМ ГРАНИЧНЫМ УСЛОВИЕМ И РАЗРЫВНЫМИ НАЧАЛЬНЫМИ ДАННЫМИ В ПЕРВОМ КВАДРАНТЕ ПЛОСКОСТИ

Аннотация. Рассматривается смешанная задача для одномерного волнового уравнения в первом квадранте плоскости с разрывными начальными и граничными условиями, возникающими в теории продольного удара по упругому полубесконечному стержню с упругим закреплением на левом конце. Предлагается новый подход к решению задачи, строящий решение как предел классических решений. Предложенный метод не использует никаких дополнительных условий, которые требовались в предыдущих методах, обеспечивая более прямое и строгое построение. Решение отыскивается в явной аналитической форме как кусочно-непрерывная функция. Для рассматриваемой задачи доказывается единственность решения и устанавливаются необходимые и достаточные условия согласования, при которых решение существует.

Ключевые слова: волновое уравнение, смешанная задача, разрывные начальные данные, динамическое граничное условие, условия согласования, условия сопряжения, продольный удар, упругий стержень

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1. Introduction and statement of the problem. In the domain $Q = (0,\infty)\times(0,\infty)$ of two independent variables $(t,x)\in\overline{Q}\subset\mathbb{R}^2$, we consider the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}\right) u(t,x) = f(t,x), \tag{1}$$

where $a > 0$, with the Cauchy conditions

$$u(0,x) = \varphi(x), \quad x \in [0,\infty),$$

$$\frac{\partial u}{\partial t}(0,x) = \psi(x) = \begin{cases} \psi_1, & x = 0, \\ \psi_2(x), & x \in (0,\infty), \end{cases} \tag{2}$$

and the boundary condition

$$\left(\frac{\partial^2}{\partial t^2} - b^2 \frac{\partial}{\partial x} + c^2\right) u(t,0) = \mu(t) = \begin{cases} \mu_1, & t = 0, \\ \mu_2(t), & t \in (0,\infty). \end{cases} \tag{3}$$

Problem (1)–(3) models the following problem from the theory of longitudinal impact [1; 2]. Suppose that an elastic, infinite, homogeneous rod of constant cross-section, whose left end $x = 0$ is elastically fastened, is subjected at the initial time $t = 0$ to an impact at the end $x = 0$ by a load that adheres to the rod. We also assume that an external volume force acts on the rod, that the displacements of the rod and their time derivatives at the initial time $t = 0$ are nonzero, and that there are no shock waves in the rod. Then, neglecting the weight of the rod and its possible vertical displacements, the displacement $u(t,x)$ of the rod satisfies problem (1)–(3), where

$$a = \sqrt{E\rho^{-1}}, \quad b = \sqrt{SEM^{-1}}, \quad c = \sqrt{kM^{-1}},$$

with $E > 0$ is Young's modulus, $\rho > 0$ is the density, $S > 0$ is the cross-sectional area, $M > 0$ is the mass of the impacting load, and $k > 0$ is the stiffness coefficient of the linear elastic element attaching the end $x = 0$. The quantity $\psi_1 - \psi_2(0+)$ is the velocity of the impacting load, and the function μ is the external force acting on the end of the rod divided by the mass of the impacting load. The value μ_1 is not arbitrary [3]; it depends on the functions f, φ, and ψ, and will be determined later. The function f is the external volume force divided by ρ.

We assume that the functions f, φ, ψ_2 and μ_2 are sufficiently smooth; specifically,

$$f \in C^1(\bar{Q}), \quad \varphi \in C^2\big([0,\infty)\big), \quad \psi_2 \in C^1\big([0,\infty)\big), \quad \mu_2 \in C\big([0,\infty)\big). \tag{4}$$

In the present paper, we will obtain the solution to the problem using the method of characteristics, which is also known for the wave equation as using D'Alembert representation of the solution that gives so-called D'Alembert type solutions. This is a fairly powerful method that is used in many recent papers on hyperbolic equations [4–9]. We also note the works related to generalized D'Alembert formulas [10–12] for equations with discontinuous conditions or coefficients and [13; 14] for equations with fractional derivatives.

Boundary condition (3) with the differential operator

$$\frac{\partial^2}{\partial t^2} - b^2 \frac{\partial}{\partial x} + c^2$$

significantly complicates the study of the equation. This makes it impossible to use reflection methods. When studying such problems using the Fourier method, a spectral parameter appears in the boundary condition [15, p. 149–150]. However, this leaves the possibility of using the characteristics method.

Note that the presence of discontinuous initial and boundary data at point (0,0) makes it impossible to directly apply classical methods. In [16], a solution to problem (1)–(3) was constructed as a limit of classical solutions of certain auxiliary mixed problems with conjugation conditions. These solutions belong to the class $C(\bar{Q}) \cap C^2(\bar{Q} \setminus \Gamma)$, where Γ is a set of measure zero consisting of points lying on the characteristics. A drawback of that approach is that some additional work, not directly related to the immediate solution of the auxiliary problems, is needed. For example, we should find Rankine – Hugoniot-type conditions. In the present paper, we propose an approach to solving problem (1)–(3) that avoids this drawback. The method constructs the solution as a limit of genuine classical solutions of the problem, i.e., solutions belonging to the class $C^2(\bar{Q})$. The approach proposed in this paper does not require any additional conditions from physics other than those used in formulating the problem.

It should be noted that the signs of the numbers b^2 and c^2 are mathematically irrelevant. Although physical considerations suggest $b^2 > 0$ and $c^2 > 0$, we treat problem (1)–(3) in a general form, regardless of the signs of b^2 and c^2. For definiteness, we assume $b^2 \in \mathbb{R}$, $c^2 \in \mathbb{R}$, $\mathrm{Re}(b) \ge 0$, $\mathrm{Im}(b) \ge 0$, $\mathrm{Re}(c) \ge 0$ and $\mathrm{Im}(c) \ge 0$.

2. Construction of the solution. Let us find a solution u of problem (1)–(3) of the following form

$$u(t,x) = \lim_{(t^*,x^*)\to(0,0)} v_{(t^*,x^*)}(t,x), \tag{5}$$

where the limit is taken pointwise, and $v_{(t^*,x^*)} \in C^2(\bar{Q})$ is a classical solution of the auxiliary problem

$$\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right) v_{(t^*,x^*)}(t,x) = f(t,x), \quad (t,x)\in Q,$$

$$v_{(t^*,x^*)}(0,x) = \varphi(x), \quad \frac{\partial v_{(t^*,x^*)}}{\partial t}(0,x) = \tilde{\psi}_{x^*}(x), \quad 0 \le x < \infty, \tag{6}$$

$$\left(\frac{\partial^2}{\partial t^2} - b^2\frac{\partial}{\partial x} + c^2\right) v_{(t^*,x^*)}(t,0) = \tilde{\mu}_{t^*}(t), \quad 0 \le t < \infty,$$

with

$$x^* > 0, \quad t^* > 0,$$

$$\tilde{\mu}_{t^*}(0) = \mu_1, \quad \tilde{\psi}_{x^*}(0) = \psi_1, \quad \tilde{\psi}_{x^*}\Big|_{[t^*,\infty)} = \psi_2\big|_{[t^*,\infty)}, \quad \tilde{\mu}_{t^*}\Big|_{[t^*,\infty)} = \mu_2\big|_{[t^*,\infty)},$$

$$\left|\tilde{\psi}_{x^*}(x)\right| \le \|\psi_2\|_{C([0,x^*])} + \left|\psi_1 - \psi_2(0)\right| \text{ for all } x \in (0,x^*),$$

$$\left|\tilde{\mu}_{t^*}(t)\right| \le \max\left\{|\mu_1|, |\mu_2(t^*)|\right\} \text{ for all } t \in (0,t^*),$$

$$\tilde{\psi}_{x^*} \in C^1\big([0,+\infty)\big), \quad \tilde{\mu}_{t^*} \in C\big([0,+\infty)\big).$$

For the functions $\tilde{\psi}_{x^*}$ and $\tilde{\mu}_{t^*}$ one can take

$$\tilde{\psi}_{x^*}(x) = \begin{cases} \psi_2(x) + \big(\psi_1 - \psi_2(0)\big)\big(1 - x/x^*\big)^2, & x \le x^*, \\ \psi_2(x), & x > x^*, \end{cases}$$

$$\tilde{\mu}_{t^*}(t) = \begin{cases} \mu_1 + \dfrac{\left(\mu_2(t^*) - \mu_1\right)t}{t^*}, & t \le t^*, \\ \mu_2(t), & t > t^*. \end{cases}$$

As it is well known [17, p. 135–136], the general solution of the wave equation is the sum of the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation. Let $w : \bar{Q} \to \mathbb{R}$ be the particular solution of the inhomogeneous equation satisfying

$$w(0,x) = \frac{\partial w}{\partial t}(0,x) = 0, \quad \frac{\partial^2 w}{\partial t^2}(0,x) = f(0,x).$$

Such a solution w exists [17, p. 138–153; 18–20] and is given by

$$w(t,x) = \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} \tilde{f}(\tau,\xi)d\xi, \tag{7}$$

where $\tilde{f}$ is an extension of f to the upper half plane:

$$\tilde{f}(t,x) = \begin{cases} f(t,0) + x\partial_x f(t,0), & x < 0, \\ f(t,x), & x \ge 0. \end{cases}$$

If $f \in C^1(\bar{Q})$, then $\tilde{f} \in C^1([0,\infty)\times\mathbb{R})$ and $w \in C^2(\bar{Q})$.

The general solution of the wave equation can then be written as:

$$v_{(t^*,x^*)}(t,x) = w(t,x) + g^{(1)}(x-at) + g^{(2)}(x+at), \tag{8}$$

where $g^{(1)}$ and $g^{(2)}$ are twice continuously differentiable functions.

Substituting (8) into the Cauchy conditions of problem (6) yields the system

$$g^{(1)}(x) + g^{(2)}(x) = \varphi(x), \quad x \in [0,\infty),$$

$$-aDg^{(1)}(x) + aDg^{(2)}(x) = \psi(x), \quad x \in [0,\infty),$$

whose solution is (see, e. g., [17, p. 139])

$$g^{(1)}(x) = \frac{\varphi(x)}{2} - \frac{1}{2a}\int_0^x \tilde{\psi}_{x^*}(\xi)d\xi + C_1, \quad x \in [0,\infty), \tag{9}$$

$$g^{(2)}(x) = \frac{\varphi(x)}{2} + \frac{1}{2a}\int_0^x \tilde{\psi}_{x^*}(\xi)d\xi - C_1, \quad x \in [0,\infty), \tag{10}$$

where C_1 is an arbitrary constant of the integration.

From representation (8) and the boundary condition of problem (6), we obtain the equality

$$\left(\frac{\partial^2}{\partial t^2} - b^2\frac{\partial}{\partial x} + c^2\right)v_{(t^*,x^*)}(t,0) = -b^2\left(\frac{\partial w}{\partial x}(t,0) + Dg^{(1)}(-at) + Dg^{(2)}(at)\right) + \frac{\partial^2 w}{\partial t^2}(t,0) +$$
$$+ c^2\left(w(t,0) + g^{(1)}(-at) + g^{(2)}(at)\right) + a^2\left(D^2 g^{(1)}(-at) + D^2 g^{(2)}(at)\right) = \tilde{\mu}_{t^*}(t), \ \ t \in (0,\infty). \tag{11}$$

Introducing the substitution $z = -at$ in this equality, we obtain the following ordinary differential equation for the function $g^{(1)}$ on the ray $(-\infty,0)$:

$$c^2\left(w\left(-\frac{z}{a},0\right)+g^{(1)}(z)+g^{(2)}(-z)\right)+a^2\left(D^2g^{(1)}(z)+a^2D^2g^{(2)}(-z)\right)-$$
$$-b^2\left(\frac{\partial w}{\partial x}\left(-\frac{z}{a},0\right)+Dg^{(1)}(z)+Dg^{(2)}(-z)\right)=\tilde{\mu}_{t^*}\left(-\frac{z}{a}\right)-\frac{\partial^2 w}{\partial t^2}\left(-\frac{z}{a},0\right),\quad z\in(-\infty,0). \tag{12}$$

For formula (8) to define a classical solution $v_{(t^*,x^*)}$, the function $g^{(1)}$ must belong to the class $C^2(\mathbb{R})$. It implies the equalities

$$\begin{gathered} g^{(1)}(0-)=\varphi^{(0)}=g^{(1)}(0+)=C_1+\frac{\varphi(0)}{2},\\ Dg^{(1)}(0-)=\psi^{(0)}=Dg^{(1)}(0+)=\frac{D\varphi(0)}{2}-\frac{\tilde{\psi}_{x^*}(0)}{2a}. \end{gathered} \tag{13}$$

We treat Eq. (12) for $g^{(1)}$, together with conditions (13), as the Cauchy problem for a second-order ordinary differential equation. Solving this problem, we obtain

$$g^{(1)}(z)=g^{(1)}_{b^4-4a^2c^2\neq 0}(z)=$$
$$=\exp\left(\frac{b^2z}{2a^2}\right)\left(\varphi^{(0)}\operatorname{ch}\left(\frac{z\sqrt{b^4-4a^2c^2}}{2a^2}\right)+\frac{2a^2\psi^{(0)}-b^2\varphi^{(0)}}{\sqrt{b^4-4a^2c^2}}\operatorname{sh}\left(\frac{z\sqrt{b^4-4a^2c^2}}{2a^2}\right)\right)+$$
$$+\int_0^z\frac{2\mathcal{M}(\xi)}{\sqrt{b^4-4a^2c^2}}\exp\left(\frac{b^2(z-\xi)}{2a^2}\right)\operatorname{sh}\left(\frac{(z-\xi)\sqrt{b^4-4a^2c^2}}{2a^2}\right)d\xi,\quad z\in(-\infty,0],\quad b^4-4a^2c^2\neq 0,$$

or

$$g^{(1)}(z)=g^{(1)}_{b^4-4a^2c^2=0}(z)=\frac{\exp\left(\frac{cz}{a}\right)\left(a(\varphi^{(0)}+z\psi^{(0)})-cz\varphi^{(0)}\right)}{a}+$$
$$+\int_0^z\frac{\exp\left(\frac{c(z-\xi)}{a}\right)(z-\xi)\mathcal{M}(\xi)}{a^2}d\xi,\quad z\in(-\infty,0],\quad b^4-4a^2c^2=0,$$

where

$$\mathcal{M}(z)=\tilde{\mu}_{t^*}\left(-\frac{z}{a}\right)-\frac{\partial^2 w}{\partial t^2}\left(-\frac{z}{a},0\right)+b^2\left(\frac{\partial w}{\partial x}\left(-\frac{z}{a},0\right)+Dg^{(2)}(-z)\right)-$$
$$-c^2\left(w\left(-\frac{z}{a},0\right)+g^{(2)}(-z)\right)-a^2D^2g^{(2)}(-z),\quad z\in(-\infty,0].$$

The expression for $g^{(1)}_{b^4-4a^2c^2\neq 0}(z)$ also remains valid when $b^4-4a^2c^2<0$, since in that case the same real solution is obtained regardless of the choice of the complex square root $\sqrt{b^4-4a^2c^2}$. If $b^4-4a^2c^2=0$, the expression for $g^{(1)}_{b^4-4a^2c^2\neq 0}(z)$ is real-valued, since in this case $c\geq 0$ by our assumptions ($b^4\geq 0$ and $a>0$). We also note that $g^{(1)}_{b^4-4a^2c^2=0}(z)=\lim\limits_{b\to\sqrt{2ac}}g^{(1)}_{b^4-4a^2c^2\neq 0}(z)$. It means that we can write the solution of Eq. (12) with conditions (13) by one formula

$$g^{(1)}(z)=G_1(z)=\exp\left(\frac{b^2z}{2a^2}\right)\left(\varphi^{(0)}\operatorname{ch}\left(\frac{z\sqrt{b^4-4a^2c^2}}{2a^2}\right)+\frac{2a^2\psi^{(0)}-b^2\varphi^{(0)}}{\sqrt{b^4-4a^2c^2}}\operatorname{sh}\left(\frac{z\sqrt{b^4-4a^2c^2}}{2a^2}\right)\right)+$$
$$+\int_0^z\frac{2\mathcal{M}(\xi)}{\sqrt{b^4-4a^2c^2}}\exp\left(\frac{b^2(z-\xi)}{2a^2}\right)\operatorname{sh}\left(\frac{(z-\xi)\sqrt{b^4-4a^2c^2}}{2a^2}\right)d\xi,\quad z\in(-\infty,0],\tag{14}$$

which in the case $b^4-4a^2c^2=0$ is understood as the limit $b\to\sqrt{2ac}$. Since

$$\mathcal{M}(\xi)=\tilde{\mu}_{t^*}\left(-\frac{\xi}{a}\right)-c^2w\left(-\frac{\xi}{a},0\right)+c^2C_1-c^2\left(\frac{1}{2a}\int_0^{-\xi}\tilde{\psi}_{x^*}(\eta)d\eta+\frac{\varphi(-\xi)}{2}\right)+$$
$$+b^2\left(\frac{\tilde{\psi}_{x^*}(-\xi)}{2a}+\frac{1}{2}\varphi'(-\xi)\right)-a^2\left(\frac{\tilde{\psi}'_{x^*}(-\xi)}{2a}+\frac{1}{2}\varphi''(-\xi)\right)+b^2\frac{\partial w}{\partial x}\left(-\frac{\xi}{a},0\right)-\frac{\partial^2 w}{\partial t^2}\left(-\frac{\xi}{a},0\right)\Bigg),\quad \xi\in(-\infty,0],$$

$$\int_0^z\frac{2c^2C_1}{\sqrt{b^4-4a^2c^2}}\exp\left(\frac{b^2(z-\xi)}{2a^2}\right)\operatorname{sh}\left(\frac{(z-\xi)\sqrt{b^4-4a^2c^2}}{2a^2}\right)d\xi=$$
$$=C_1\left(1-\exp\left(\frac{b^2z}{2a^2}\right)\operatorname{ch}\left(\frac{z\sqrt{b^4-4a^2c^2}}{2a^2}\right)+\frac{b^2}{\sqrt{b^4-4a^2c^2}}\exp\left(\frac{b^2z}{2a^2}\right)\operatorname{sh}\left(\frac{z\sqrt{b^4-4a^2c^2}}{2a^2}\right)\right),$$

we can write

$$g^{(1)}(z)=G_1(z)\big|_{C_1=0}+C_1,\quad z\in(-\infty,0].\tag{15}$$

Under the smoothness conditions

$$f\in C^1(\bar{Q}),\quad \varphi\in C^2([0,\infty)),\quad \tilde{\psi}_{x^*}\in C^1([0,\infty)),\quad \tilde{\mu}_{t^*}\in C([0,\infty)),\tag{16}$$

representation (8), together with relations (9), (10), and (15), defines a function $v_{(t^*,x^*)}\in C^2(\bar{Q})$ if and only if

$$D^kg^{(1)}(0-)=D^kg^{(1)}(0+),\quad k=0,1,2,\tag{17}$$

since $g^{(1)}\in C^2([0,\infty))$, $g^{(1)}\in C^2((-\infty,0])$, and $g^{(2)}\in C^2([0,\infty))$ [17, p. 137–138]. Condition (17) is met for $k=0$ and $k=1$ by construction. By the direct calculation we get

$$D^2g^{(1)}(0-)-D^2g^{(1)}(0+)=-\frac{c^2w(0,0)}{a^2}+\frac{\mu(0)}{a^2}-\frac{c^2\varphi(0)}{a^2}+\frac{b^2\varphi'(0)}{a^2}-\varphi''(0)+$$
$$+\frac{b^2\partial_xw(0,0)}{a^2}-\frac{\partial_t^2w(0,0)}{a^2},$$

which, by the properties of the function w, is equivalent to

$$D^2g^{(1)}(0-)-D^2g^{(1)}(0+)=a^2\left(\tilde{\mu}_{t^*}(0)-f(0,0)-c^2\varphi(0)+b^2\varphi'(0)-a^2\varphi''(0)\right).\tag{18}$$

Hence, condition (17) is satisfied if and only if

$$\tilde{\mu}_{t^*}(0)-f(0,0)-c^2\varphi(0)+b^2\varphi'(0)-a^2\varphi''(0)=0.\tag{19}$$

Combining (8)–(10) and (15), we obtain

$$v_{(t^*,x^*)}(t,x)=w(t,x)+\frac{\varphi(x-at)+\varphi(x+at)}{2}+\frac{1}{2a}\int\limits_{x-at}^{x+at}\tilde{\psi}(\xi)d\xi,\quad t\ge 0,\quad x\ge 0,\quad x-at\ge 0,\tag{20}$$

$$v_{(t^*,x^*)}(t,x)=w(t,x)+\frac{1}{2a}\int\limits_{0}^{x+at}\tilde{\psi}(\eta)d\eta+\frac{\varphi(x+at)}{2}+\exp\left(\frac{b^2(x-at)}{2a^2}\right)\times$$
$$\times\left(\frac{\varphi(0)}{2}\operatorname{ch}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)+\frac{\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)\left(2a^2\left(\frac{\varphi'(0)}{2}-\frac{\tilde{\psi}_{x^*}(0)}{2a}\right)-\frac{b^2\varphi(0)}{2}\right)}{\sqrt{b^4-4a^2c^2}}\right)+$$
$$+\int\limits_{0}^{x-at}\frac{2\exp\left(\frac{b^2(x-at-\xi)}{2a^2}\right)}{\sqrt{b^4-4a^2c^2}}\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at-\xi)}{2a^2}\right)\times$$
$$\times\left(\tilde{\mu}\left(-\frac{\xi}{a}\right)-c^2w\left(-\frac{\xi}{a},0\right)-c^2\left(\frac{1}{2a}\int\limits_{0}^{-\xi}\tilde{\psi}_{x^*}(\eta)d\eta+\frac{\varphi(-\xi)}{2}\right)+\right.$$
$$\left.+b^2\left(\frac{\tilde{\psi}_{x^*}(-\xi)}{2a}+\frac{1}{2}\varphi'(-\xi)\right)-a^2\left(\frac{\tilde{\psi}'_{x^*}(-\xi)}{2a}+\frac{1}{2}\varphi''(-\xi)\right)+b^2\frac{\partial w}{\partial x}\left(-\frac{\xi}{a},0\right)-\frac{\partial^2 w}{\partial t^2}\left(-\frac{\xi}{a},0\right)\right)d\xi,$$
$$t\ge 0,\quad x\ge 0,\quad x-at\le 0.\tag{21}$$

Note that in the case $b^4-4a^2c^2=0,$ the last formula is understood as the limit $b\to\sqrt{2ac}.$

We now prove the uniqueness of the solution $v_{(t^*,x^*)}\in C^2(\bar{Q})$ to problem (6). Suppose problem (6) has two classical solutions, $v^{(1)}_{(t^*,x^*)}$ and $v^{(2)}_{(t^*,x^*)}$. Then their difference $V=v^{(1)}_{(t^*,x^*)}-v^{(2)}_{(t^*,x^*)}$ satisfies the problem

$$\begin{gathered}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)V(t,x)=0,\quad (t,x)\in Q,\\ V(0,x)=\frac{\partial V}{\partial t}(0,x)=0,\quad 0\le x<\infty,\\ \left(\frac{\partial^2}{\partial t^2}-b^2\frac{\partial}{\partial x}+c^2\right)V(t,0)=0,\quad 0\le t<\infty.\end{gathered}\tag{22}$$

From formulas (20) and (21), it follows that problem (22) has only the trivial solution $V\equiv 0.$ This proves uniqueness.

Thus, we have proved the following lemma.

L e m m a . *Let conditions* (16) *be satisfied. Problem* (6) *has a unique solution* $v_{(t^*,x^*)}$ *in the class* $C^2(\bar{Q})$ *if and only if the matching condition* (19) *holds. This solution is given by formulas* (7)*,* (20)*, and* (21)*.*

We now turn to problem (1)–(3). To pass to the limit in (4) with (20) and (21) we use the following statements based on dominated convergence theorem.

S t a t e m e n t 1. *Assume* $l\in(0,\infty)$ *and* $K\in C([0,l])$*. Then*

$$\lim_{x^*\to 0}\int_0^l K(\xi)\tilde{\psi}_{x^*}(\xi)\,d\xi=\int_0^l K(\xi)\psi_2(\xi)\,d\xi.$$

P r o o f. The functions $\tilde{\psi}_{x^*}$ converge pointwise to the function ψ, so the functions $K\tilde{\psi}_{x^*}$ converge pointwise to the function $K\psi$ too. There holds the estimate

$$\left|\tilde{\psi}_{x^*}(x)\right|\le\left\|\psi_2\right\|_{C([0,l])}+\left|\psi_1-\psi_2(0)\right|,\quad x\in[0,l],$$

due to the requirements imposed on the function $\tilde{\psi}_{x^*}$ in Definition 1. Thus, the number $\left(\left\|\psi_2\right\|_{C([0,l])}+\left|\psi_1-\psi_2(0)\right|\right)\left\|K\right\|_{C([0,l])}$ uniformly bounds the family of the functions $K\tilde{\psi}_{x^*}$ on the range $[0,l]$. It means that we can apply the dominated convergence theorem (or the bounded convergence theorem) and get

$$\lim_{x^*\to 0}\int_0^l K(\xi)\tilde{\psi}_{x^*}(\xi)\,d\xi=\int_0^l K(\xi)\left(\lim_{x^*\to 0}\tilde{\psi}_{x^*}(\xi)\right)d\xi=\int_0^l K(\xi)\psi(\xi)\,d\xi.$$

Since $\psi(x)=\psi_2(x)$ almost everywhere on the set $[0,l]$, we obtain

$$\lim_{x^*\to 0}\int_0^l K(\xi)\tilde{\psi}_{x^*}(\xi)\,d\xi=\int_0^l K(\xi)\psi(\xi)\,d\xi=\int_0^l K(\xi)\psi_2(\xi)\,d\xi.$$

The statement is proved.

S t a t e m e n t 2. *Assume* $l\in(0,\infty)$ *and* $K\in C([0,l])$. *Then*

$$\lim_{t^*\to 0}\int_0^l K(\xi)\mu_{t^*}(\xi)\,d\xi=\int_0^l K(\xi)\mu_2(\xi)\,d\xi.$$

S t a t e m e n t 3. *Assume* $l\in(0,\infty)$ *and* $K\in C([0,l])$. *Then*

$$\lim_{x^*\to 0}\int_0^l K(\xi)\left(\int_0^\xi \tilde{\psi}_{x^*}(\eta)d\eta\right)d\xi=\int_0^l K(\xi)\left(\int_0^\xi \psi_2(\eta)d\eta\right)d\xi.$$

The p r o o f s of Statements 2 and 3 are similar to the proof of Statement 1.

Note that we cannot directly pass to the limit in (5) due to the presence of the term

$$\int_0^{x-at}-\frac{a^2\exp\left(\dfrac{b^2(x-at-\xi)}{2a^2}\right)}{\sqrt{b^4-4a^2c^2}}\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}\,(x-at-\xi)}{2a^2}\right)\tilde{\psi}'_{x^*}(-\xi)\,d\xi,$$

where the integrand cannot be dominated. However, this term can be written as

$$\int_0^{x-at}-\frac{a^2\exp\left(\dfrac{b^2(x-at-\xi)}{2a^2}\right)}{\sqrt{b^4-4a^2c^2}}\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}\,(x-at-\xi)}{2a^2}\right)\tilde{\psi}'_{x^*}(-\xi)\,d\xi=$$

$$=-\frac{a\exp\left(\dfrac{b^2(x-at)}{2a^2}\right)\operatorname{sh}\left(\dfrac{\sqrt{b^4-4a^2c^2}\,(x-at)}{2a^2}\right)\tilde{\psi}_{x^*}(0)}{\sqrt{b^4-4a^2c^2}}+\int_0^{x-at}\frac{\exp\left(\dfrac{b^2(x-at-\xi)}{2a^2}\right)\tilde{\psi}_{x^*}(-\xi)}{2a\sqrt{b^4-4a^2c^2}}\times$$

$$\times\left(b^2\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(at-x+\xi)}{2a^2}\right)+\sqrt{b^4-4a^2c^2}\operatorname{ch}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at-\xi)}{2a^2}\right)\right)d\xi.$$

This implies

$$v_{(t^*,x^*)}(t,x)=w(t,x)+\frac{1}{2a}\int_0^{x+at}\tilde{\psi}_{x^*}(\eta)d\eta+\frac{\varphi(x+at)}{2}+\exp\left(\frac{b^2(x-at)}{2a^2}\right)\times$$

$$\times\left(\frac{\varphi(0)}{2}\operatorname{ch}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)+\frac{\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)\left(a^2\varphi'(0)-2a\tilde{\psi}_{x^*}(0)-\frac{b^2\varphi(0)}{2}\right)}{\sqrt{b^4-4a^2c^2}}\right)+$$

$$+\int_0^{x-at}\frac{2\exp\left(\frac{b^2(x-at-\xi)}{2a^2}\right)}{\sqrt{b^4-4a^2c^2}}\left\{\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at-\xi)}{2a^2}\right)\times\right.$$

$$\times\left(\tilde{\mu}_{t^*}\left(-\frac{\xi}{a}\right)-c^2w\left(-\frac{\xi}{a},0\right)-c^2\left(\frac{1}{2a}\int_0^{-\xi}\tilde{\psi}_{x^*}(\eta)\mathrm{d}\eta+\frac{\varphi(-\xi)}{2}\right)+\right.$$

$$\left.+b^2\left(\frac{3\tilde{\psi}_{x^*}(-\xi)}{4a}+\frac{1}{2}\varphi'(-\xi)\right)-\frac{a^2}{2}\varphi''(-\xi)+b^2\frac{\partial w}{\partial x}\left(-\frac{\xi}{a},0\right)-\frac{\partial^2 w}{\partial t^2}\left(-\frac{\xi}{a},0\right)\right)+$$

$$\left.+\frac{\tilde{\psi}_{x^*}(-\xi)}{4a}\sqrt{b^4-4a^2c^2}\operatorname{ch}\left(\frac{\sqrt{b^4-4a^2c^2}(at-x+\xi)}{2a^2}\right)\right\}d\xi,\quad t\ge 0,\quad x\ge 0,\quad x-at\le 0. \qquad (23)$$

Statements 1–3 applied to expression (4) with (20) and (21) allow us to write

$$u(t,x)=u^{(1)}(t,x)=w(t,x)+\frac{\varphi(x-at)+\varphi(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi_2(\xi)d\xi,$$
$$t\ge 0,\quad x\ge 0,\quad x-at\ge 0, \qquad (24)$$

$$u(t,x)=u^{(2)}(t,x)=w(t,x)+\frac{1}{2a}\int_0^{x+at}\psi_2(\eta)d\eta+\frac{\varphi(x+at)}{2}+\exp\left(\frac{b^2(x-at)}{2a^2}\right)\times$$

$$\times\left(\frac{\varphi(0)}{2}\operatorname{ch}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)+\frac{\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)\left(a^2\varphi'(0)-2a\psi_1-\frac{b^2\varphi(0)}{2}\right)}{\sqrt{b^4-4a^2c^2}}\right)+$$

$$+\int_0^{x-at}\frac{2\exp\left(\frac{b^2(x-at-\xi)}{2a^2}\right)}{\sqrt{b^4-4a^2c^2}}\left\{\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at-\xi)}{2a^2}\right)\times\right.$$

$$\times\left(\mu_2\left(-\frac{\xi}{a}\right)-c^2w\left(-\frac{\xi}{a},0\right)-c^2\left(\frac{1}{2a}\int_0^{-\xi}\psi_2(\eta)\mathrm{d}\eta+\frac{\varphi(-\xi)}{2}\right)+\right.$$

$$+b^2\left(\frac{3\psi_2(-\xi)}{4a}+\frac{1}{2}\varphi'(-\xi)\right)-\frac{a^2}{2}\varphi''(-\xi)+b^2\frac{\partial w}{\partial x}\left(-\frac{\xi}{a},0\right)-\left.\frac{\partial^2 w}{\partial t^2}\left(-\frac{\xi}{a},0\right)\right)+$$

$$\left.+\frac{\psi_2(-\xi)}{4a}\sqrt{b^4-4a^2c^2}\,\mathrm{ch}\left(\frac{\sqrt{b^4-4a^2c^2}\,(at-x+\xi)}{2a^2}\right)\right\}d\xi,\quad t\ge 0,\quad x\ge 0,\quad x-at\le 0. \tag{25}$$

Representations (24) and (25) hold only under the condition

$$\mu_1=f(0,0)+c^2\varphi(0)+a^2D^2\varphi(0)-b^2D\varphi(0) \tag{26}$$

since the functions $v_{(t^*,x^*)}$ are classical solutions and do not exist if the matching condition (19) is violated. Note that equality (26) holds if and only if condition (19) is satisfied, since $\tilde{\mu}_{t^*}(0)=\mu_1$.

3. Properties of the constructed solution. The conjugation conditions on the characteristic $x-at=0$ for the solution u of problem (1)–(3) can be written explicitly as

$$[(u)^+-(u)^-](t,at)=0, \tag{27}$$

$$\left[\left(\frac{\partial u}{\partial x}\right)^+-\left(\frac{\partial u}{\partial x}\right)^-\right](t,at)=\frac{\psi_1-\psi_2(0+)}{a}, \tag{28}$$

$$\left[\left(\frac{\partial u}{\partial t}\right)^+-\left(\frac{\partial u}{\partial t}\right)^-\right](t,at)=\psi_2(0+)-\psi_1, \tag{29}$$

$$\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^+-\left(\frac{\partial^2 u}{\partial x^2}\right)^-\right](t,at)=$$

$$=\frac{b^2\left(\psi_1-\psi_2(0+)-a\varphi'(0)\right)+a\left(f(0,0)-\mu_2(0+)+c^2\varphi(0)+a^2\varphi''(0)\right)}{a^3}, \tag{30}$$

$$\left[\left(\frac{\partial^2 u}{\partial t\partial x}\right)^+-\left(\frac{\partial^2 u}{\partial t\partial x}\right)^-\right](t,at)=$$

$$=\frac{b^2\left(\psi_2(0+)-\psi_1+a\varphi'(0)\right)-a\left(f(0,0)-\mu_2(0+)+c^2\varphi(0)+a^2\varphi''(0)\right)}{a^2}, \tag{31}$$

$$\left[\left(\frac{\partial^2 u}{\partial t^2}\right)^+-\left(\frac{\partial^2 u}{\partial t^2}\right)^-\right](t,at)=$$

$$=f(0,0)-\mu_2(0+)+c^2\varphi(0)-\frac{b^2\left(\psi_2(0+)-\psi_1+a\varphi'(0)\right)}{a}+a^2\varphi''(0). \tag{32}$$

Here $(h)^{\pm}$ denotes the limiting values of the function h on the two sides of the characteristic $x-at=0$; that is,

$$(h)^{\pm}(t, x = at) = \lim_{\delta \to 0+} h(t, at \pm \delta).$$

Formulas (27) and (28) show that the solution u is consistent with physical conservation laws [21, p. 71]. For the proof, we refer the reader to [16].

Under assumptions (4), the function u belongs to the classes $C^2\left(\overline{\tilde{Q}^{(1)}}\right)$ and $C^2\left(\overline{\tilde{Q}^{(2)}}\right)$, where

$$\tilde{Q}^{(1)} = \left\{(t, x) : t \ge 0, x \ge 0, x - at \ge 0\right\},$$

$$\tilde{Q}^{(2)} = \left\{(t, x) : t \ge 0, x \ge 0, x - at \le 0\right\}.$$

Moreover, from (27) it follows that $u \in C(\overline{Q})$.

The question arises whether the constructed function u satisfies the wave equation (1) and conditions (2) and (3). By the direct calculation, we establish that the function u satisfies Equation (1) on the set $Q \setminus \{(t, x) : x = at\}$, the first Cauchy condition (2) on the set $[0,\infty)$, the second Cauchy condition (2) on the set $(0,\infty)$, and the boundary condition (3) on the set $(0,\infty)$. On the one hand, since the quantities $\partial_t u(0,0)$ and $\partial_x u(0,0)$ are not defined in the usual sense, we cannot say that the second Cauchy condition (2) and the boundary condition (3) hold on the closed ray $[0,\infty)$ in the classical sense. But, on the other hand, we have

$$\lim_{\substack{(t, x) \to (0, 0) \\ x - at > 0}} \partial_t u(t, x) = \psi_2(0), \qquad \lim_{\substack{(t, x) \to (0, 0), \\ x - at < 0}} \partial_t u(t, x) = \psi_1.$$

We can show that conditions (2) and (3) are fulfilled on the entire half-line $[0,\infty)$ in the limit sense. To do this, we introduce the operator

$$\frac{\tilde{\partial}^{k+p}}{\tilde{\partial} t^k \tilde{\partial} x^p}[u](t, x) := \lim_{(t^*, x^*) \to (0, 0)} \frac{\partial^{k+p}}{\partial t^k \partial x^p} v_{(t^*, x^*)}(t, x).$$

We have

$$\frac{\tilde{\partial} u}{\tilde{\partial} t}(0, x) = \psi(x) = \begin{cases} \psi_1, & x = 0, \\ \psi_2(x), & x \in (0, \infty), \end{cases}$$

$$\left(\frac{\tilde{\partial}^2}{\tilde{\partial} t^2} - b^2 \frac{\tilde{\partial}}{\tilde{\partial} x} + c^2\right) u(t, 0) = \mu(t) = \begin{cases} \mu_1, & t = 0, \\ \mu_2(t), & t \in (0, \infty). \end{cases}$$

Moreover, it can be shown that the function u satisfies Equation (1) in the sense of distributions on the set Q. Let us prove that: since $v_{(t^*, x^*)}$ is a classical solution of the wave equation, the function $v_{(t^*, x^*)}$ solves this equation in the distributional sense, i. e.,

$$\iint_Q v_{(t^*, x^*)}(t, x)\left(\frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}\right) y(t, x)\,dt\,dx = \iint_Q y(t, x) f(t, x)\,dt\,dx$$

for all $y \in C^\infty(Q)$. Denote $K = \operatorname{supp}(y) \subset Q$. Due to the restrictions imposed on the functions φ, $\tilde{\psi}_{x^*}$, $\tilde{\mu}_{t^*}$ and f, the family of functions $v_{(t^*, x^*)}$ is uniformly bounded on the compact set K. Thus, we can use the dominated convergence theorem to write

$$\lim_{(t^*, x^*) \to (0,0)} \iint_Q v_{(t^*, x^*)}(t, x)\left(\frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}\right) y(t, x)\,dt\,dx =$$

$$=\iint_Q \lim_{(t^*,x^*)\to(0,0)}\left(v_{(t^*,x^*)}(t,x)\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)y(t,x)\right)dtdx=\iint_Q u(t,x)\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)y(t,x)dtdx.$$

Since

$$\lim_{(t^*,x^*)\to(0,0)}\iint_Q v_{(t^*,x^*)}(t,x)\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)y(t,x)dtdx=\iint_Q y(t,x)f(t,x)dtdx,$$

due to the fact that $v_{(t^*,x^*)}$ solves the wave equation in the classical sense, we obtain

$$\iint_Q u(t,x)\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)y(t,x)dtdx=\iint_Q y(t,x)f(t,x)dtdx,$$

for arbitrary $y\in C^\infty(Q)$. The last equality means that the function u solves Equation (1) as a distribution.

Note that in the case $b^4-4a^2c^2<0$ the behavior of the constructed solution u is oscillatory, unlike the case $b^4-4a^2c^2\ge 0$.

4. Classical solution with conjugation conditions. The solution u constructed in Sec. 2 can be regarded as a classical solution of Eq. (1) with initial conditions (2), boundary condition (3), and certain additional conjugation conditions. However, since the quantities $\partial_t u(0,0)$ and $\partial_x u(0,0)$ are not defined in the usual sense, we cannot require that conditions (2) and (3) hold on the closed ray $[0,\infty)$ in the classical sense. Consequently, the number μ_1 cannot be determined within this framework and must be postulated. Moreover, formulas (27)–(32), which determine the solution u, do not contain the quantity μ_1.

Problem (1)–(3) with conjugation conditions on the characteristic. Find a classical solution of Eq. (1) on the set $Q\setminus\{(t,x):x=at\}$ satisfying the Cauchy conditions (2) on the set $(0,\infty)$, the boundary condition (3) on the set $(0,\infty)$, and the conjugation conditions (27) and (28).

Note that in this formulation, μ_1 does not appear.

Under conditions (4), the existence of a classical solution u of problem (1)–(3) with conjugation conditions on the characteristic follows from formulas (24) and (25). Indeed, a direct calculation shows that the function u defined by formulas (24) and (25) satisfies Eq. (1) on the set $Q\setminus\{(t,x):x=at\}$ and conditions (2), (3), (27), and (28) on the set $(0,\infty)$.

We now prove the uniqueness of the solution u of problem (1)–(3) with conjugation conditions. We use the fact that this solution consists of classical solutions of two subproblems: the Cauchy problem on the set $\overline{\tilde{Q}^{(1)}}$ and the Picard problem on the set $\overline{\tilde{Q}^{(2)}}$. Consequently, the uniqueness of the classical solution u of problem (1)–(3) with conjugation conditions follows from the uniqueness of the solutions of these subproblems (see [17, p. 138–142] for the Cauchy problem and [6] for the Picard problem). Note that the proof in [6] is given for a more general case: a differential operator of the n^{th} order with variable coefficients in the boundary condition. In the present paper we have the second-order differential operator with constant coefficients in condition (2). Therefore, we can give a simpler proof than one presented in [6]. So, we need to establish that the problem

$$\begin{gathered}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)v(t,x)=f(t,x),\quad (t,x)\in\tilde{Q}^{(1)},\\ v(t,x=at)=\gamma_1(t),\quad \frac{\partial v}{\partial x}(t,x=at)=\gamma_2(t),\\ t\in[0,\infty),\quad \left(\frac{\partial^2}{\partial t^2}-b^2\frac{\partial}{\partial x}+c^2\right)v(t,0)=\gamma_3(t),\quad t\in[0,\infty),\end{gathered}\tag{33}$$

cannot have more than one classical solution.

Suppose problem (33) has two classical solutions, $v^{(1)}$ and $v^{(2)}$. Then their difference $V = v^{(1)} - v^{(2)}$ satisfies the problem

$$\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)V(t,x) = 0, \quad (t,x) \in \tilde{Q}^{(1)}, \tag{34}$$

$$\left(\frac{\partial^2}{\partial t^2} - b^2\frac{\partial}{\partial x} + c^2\right)V(t,0) = V(t, x = at) = \frac{\partial V}{\partial x}(t, x = at) = 0, \quad t \in [0,\infty). \tag{35}$$

The general solution of Eq. (34) is

$$V(t,x) = p_1(x - at) + p_2(x + at), \tag{36}$$

where $p_1 \in C^2((-\infty,0])$ and $p_2 \in C^2([0,\infty))$. Substituting (36) into conditions (35) yields

$$\begin{gathered} p_1(0) + p_2(2at) = 0, \quad p_1'(0) + p_2'(2at) = 0, \\ c^2\left(p_1(-at) + p_2(at)\right) - b^2\left(p_1'(-at) + p_2'(at)\right) + a^2 p_1''(-at) + a^2 p_2''(at) = 0, \end{gathered} \tag{37}$$

for all $t \in [0,\infty)$. The first equation of system (37) implies

$$p_2(z) = -p_1(0), \quad z \in [0,\infty). \tag{38}$$

Substituting (38) into the last two equations of system (37) yields

$$p_1'(0) = 0, \tag{39}$$

$$c^2\left(p_1(z) - p_1(0)\right) - b^2 p_1'(z) + a^2 p_1''(z) = 0, \quad z \in (-\infty,0]. \tag{40}$$

Treating $p_1(0)$ as a constant independent of $p_1(z)$, we find the solution of Eq. (40) in the implicit analytic form

$$\begin{gathered} p(z) = p_1(0) + \exp\left(\frac{1}{2}\left(\frac{b^2}{a^2} - \frac{\sqrt{b^4 - 4a^2c^2}}{a^2}\right)z\right)C^{(1)} + \\ + \exp\left(\frac{1}{2}\left(\frac{b^2}{a^2} + \frac{\sqrt{b^4 - 4a^2c^2}}{a^2}\right)z\right)C^{(2)}, \quad z \in (-\infty,0], \quad b^4 - 4a^2c^2 \neq 0, \end{gathered} \tag{41}$$

$$p(z) = p_1(0) + \exp\left(\frac{cz}{a}\right)C^{(1)} + \exp\left(\frac{cz}{a}\right)zC^{(2)}, \quad z \in (-\infty,0], \quad b^4 - 4a^2c^2 = 0. \tag{42}$$

We substitute $z = 0$ into expressions (41) and (42) to obtain

$$p(0) = p_1(0) + C^{(1)} + C^{(2)}, \quad b^4 - 4a^2c^2 \neq 0,$$

$$p(0) = p_1(0) + C^{(1)}, \quad b^4 - 4a^2c^2 = 0,$$

which produces equations

$$C^{(1)} + C^{(2)} = 0, \quad b^4 - 4a^2c^2 \neq 0, \tag{43}$$

$$C^{(1)} = 0, \quad b^4 - 4a^2c^2 = 0. \tag{44}$$

Due to relations (41) and (42), condition (39) implies

$$\frac{1}{2}\left(\frac{b^2}{a^2}-\frac{\sqrt{b^4-4a^2c^2}}{a^2}\right)C^{(1)}+\frac{1}{2}\left(\frac{b^2}{a^2}+\frac{\sqrt{b^4-4a^2c^2}}{a^2}\right)C^{(2)}=0,\quad b^4-4a^2c^2\neq 0, \tag{45}$$

$$\frac{cC^{(1)}}{a}+C^{(2)}=0,\quad b^4-4a^2c^2=0. \tag{46}$$

System (43) and (45) has a unique solution $C^{(1)} = C^{(2)} = 0$ since

$$\det\begin{pmatrix} 1 & 1 \\ \frac{1}{2}\left(\frac{b^2}{a^2}-\frac{\sqrt{b^4-4a^2c^2}}{a^2}\right) & \frac{1}{2}\left(\frac{b^2}{a^2}+\frac{\sqrt{b^4-4a^2c^2}}{a^2}\right)\end{pmatrix}=\frac{\sqrt{b^4-4a^2c^2}}{a^2}\neq 0.$$

System (44) and (46) admits only one solution $C^{(1)} = C^{(2)} = 0$ too. Thus, the solution to Eq. (37) can be represented as

$$p_1(-z)=p_1(0),\quad p_2(z)=-p_1(0),\quad z\in[0,\infty). \tag{47}$$

From (47) it follows that system (37) has a solution $p_1(-z)=\alpha,\ p_2(z)=-\alpha,\ z\in[0,\infty)$, where α is an arbitrary real constant. Then, according to formula (36), $V\equiv 0$, which means $v^{(1)}\equiv v^{(2)}$ and the uniqueness of the Picard subproblem is proved.

Note that the solution of problem (1)–(3) with the conjugation conditions (27) and (28) also satisfy the conjugation conditions (29)–(32).

Thus, we have proved the following theorem.

T h e o r e m 1. *Let conditions* (4) *be satisfied. Problem* (1)–(3) *with the conjugation conditions* (27) *and* (28) *has a unique classical solution u, given by formulas* (24) *and* (25)*, and belonging to the classes* $C(\bar{Q})$, $C^2\left(\overline{\tilde{Q}^{(1)}}\right)$, *and* $C^2\left(\overline{\tilde{Q}^{(2)}}\right)$.

5. Piecewise smooth solution. Based on Sec. 4, we can introduce the following definition.

D e f i n i t i o n 1. Assume that Eq. (26) holds. We call the function u a piecewise smooth solution of problem (1)–(3) if it is a classical solution of problem (1)–(3) with the conjugation conditions (27) and (28).

The following theorem holds.

T h e o r e m 2. *Let conditions* (4) *and* (26) *be satisfied. Problem* (1)–(3) *has a unique piecewise smooth solution u, given by formulas* (24) *and* (25)*, and belonging to the classes* $C(\bar{Q})$, $C^2\left(\overline{\tilde{Q}^{(1)}}\right)$, *and* $C^2\left(\overline{\tilde{Q}^{(2)}}\right)$.

The p r o o f is a direct consequence of the preceding arguments.

6. Generalized solution. There is another approach to constructing the solution of a mixed problem (see, e. g., [22, p. 19]). This approach is based on the fact that the formulas defining a sufficiently good solution remain meaningful even when the data of the mixed problem are not smooth enough to guarantee the existence of this solution.

Indeed, consider formulas (24) and (25), which, by Theorem 2, determine the piecewise smooth solution of problem (1)–(3) under the conditions

$$f\in C^1(\bar{Q}),\quad \varphi\in C^2([0,\infty)),\quad \psi_2\in C^1([0,\infty)),\quad \mu\in C([0,\infty)), \tag{48}$$

$$\mu_1=\mu(0)=f(0,0)+c^2\varphi(0)+a^2D^2\varphi(0)-b^2D\varphi(0). \tag{49}$$

By integrating by parts twice we rewrite formula (25) in the following form

$$u(t,x)=w(t,x)+\frac{1}{2a}\int_0^{x+at}\psi_2(\eta)d\eta+\frac{\varphi(x+at)}{2}+$$
$$+\exp\left(\frac{b^2(x-at)}{2a^2}\right)\left(\varphi(0)\operatorname{ch}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)+\right.$$
$$\left.+\frac{\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)\left(b^2\varphi(0)-2a\psi_1\right)}{\sqrt{b^4-4a^2c^2}}\right)-\frac{\varphi(at-x)}{2}+$$
$$+\int_0^{x-at}\frac{2\exp\left(\frac{b^2(x-at-\xi)}{2a^2}\right)}{\sqrt{b^4-4a^2c^2}}\left\{\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at-\xi)}{2a^2}\right)\times\right.$$
$$\times\left(\mu_2\left(-\frac{\xi}{a}\right)-c^2w\left(-\frac{\xi}{a},0\right)-\frac{c^2}{2a}\int_0^{-\xi}\psi_2(\eta)d\eta+\right.$$
$$\left.+\frac{3b^2}{4a}\psi_2(-\xi)-\frac{b^4}{2a^2}\varphi(-\xi)+b^2\frac{\partial w}{\partial x}\left(-\frac{\xi}{a},0\right)-\frac{\partial^2 w}{\partial t^2}\left(-\frac{\xi}{a},0\right)\right)+\sqrt{b^4-4a^2c^2}\times$$
$$\left.\times\operatorname{ch}\left(\frac{\sqrt{b^4-4a^2c^2}(at-x+\xi)}{2a^2}\right)\left(\frac{\psi_2(-\xi)}{4a}-\frac{b^2\varphi(-\xi)}{2a^2}\right)\right\}d\xi,\quad t\ge 0,\quad x\ge 0,\quad x-at\le 0. \tag{50}$$

However, formulas (24) and (50) remain meaningful under weaker conditions, for example,

$$f\in C^1(\overline{Q}),\quad \varphi\in C\big([0,\infty)\big),\quad \psi_2\in C\big([0,\infty)\big),\quad \mu_2\in C\big([0,\infty)\big). \tag{51}$$

This allows us to introduce the following definition.

D e f i n i t i o n 2. Let conditions (51) be satisfied. Following [22, p. 19], we call the function u a generalized solution of problem (1)–(3) if it is defined by formulas (24) and (50).

Definition 2 of a generalized solution is based on the following property. Assume that conditions (51) hold and the function u is a generalized solution to problem (1)–(3). Let there exist the sequences $(\varphi^{(n)})$ and $\left(\psi_2^{(n)}\right)$ such that $\varphi^{(n)}\in C^2\big([0,\infty)\big)$ for all $n\in\mathbb{N}$, $\psi_2^{(n)}\in C^1\big([0,\infty)\big)$ for all $n\in\mathbb{N}$, $(\varphi^{(n)})$ converges uniformly to φ, and $\left(\psi_2^{(n)}\right)$ converges uniformly to ψ_2 and the equality

$$\mu_1=f(0,0)+c^2\varphi^{(n)}(0)+a^2D^2\varphi^{(n)}(0)-b^2D\varphi^{(n)}(0)$$

holds for all $n\in\mathbb{N}$. Let u_n be the solution to problem

$$\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u_n(t,x)=f(t,x),$$

$$u(0,x)=\varphi^{(n)}(x),\quad x\in[0,\infty),\quad \frac{\partial u_n}{\partial t}(0,x)=\begin{cases}\psi_1, & x=0,\\ \psi_2^{(n)}(x), & x\in(0,\infty),\end{cases}$$

$$\left(\frac{\partial^2}{\partial t^2}-b^2\frac{\partial}{\partial x}+c^2\right)u(t,0)=\mu(t)=\begin{cases}\mu_1, & t=0,\\ \mu_2(t), & t\in(0,\infty),\end{cases}$$

in the sense of Definition 1. Then the sequence (u_n) converges pointwise to the generalized solution u. To prove it we pass to the limit in the formulas

$$u_n(t,x) = w(t,x) + \frac{\varphi^{(n)}(x-at)+\varphi^{(n)}(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi_2^{(n)}(\xi)d\xi, \quad t\ge 0,\quad x\ge 0,\quad x-at\ge 0,$$

$$u_n(t,x) = w(t,x) + \frac{1}{2a}\int_0^{x+at}\psi_2^{(n)}(\eta)d\eta + \frac{\varphi^{(n)}(x+at)}{2} +$$
$$+\exp\left(\frac{b^2(x-at)}{2a^2}\right)\left(\varphi^{(n)}(0)\operatorname{ch}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)+\right.$$
$$\left.+\frac{\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at)}{2a^2}\right)\left(b^2\varphi^{(n)}(0)-2a\psi_1\right)}{\sqrt{b^4-4a^2c^2}}\right)-\frac{\varphi^{(n)}(at-x)}{2}+$$
$$+\int_0^{x-at}\frac{2\exp\left(\frac{b^2(x-at-\xi)}{2a^2}\right)}{\sqrt{b^4-4a^2c^2}}\left\{\operatorname{sh}\left(\frac{\sqrt{b^4-4a^2c^2}(x-at-\xi)}{2a^2}\right)\times\right.$$
$$\times\left(\mu_2\left(-\frac{\xi}{a}\right)-c^2w\left(-\frac{\xi}{a},0\right)-\frac{c^2}{2a}\int_0^{-\xi}\psi_2^{(n)}(\eta)d\eta+\right.$$
$$\left.+\frac{3b^2}{4a}\psi_2^{(n)}(-\xi)-\frac{b^4}{2a^2}\varphi^{(n)}(-\xi)+b^2\frac{\partial w}{\partial x}\left(-\frac{\xi}{a},0\right)-\frac{\partial^2 w}{\partial t^2}\left(-\frac{\xi}{a},0\right)\right)+\sqrt{b^4-4a^2c^2}\times$$
$$\left.\times\operatorname{ch}\left(\frac{\sqrt{b^4-4a^2c^2}(at-x+\xi)}{2a^2}\right)\left(\frac{\psi_2^{(n)}(-\xi)}{4a}-\frac{b^2\varphi^{(n)}(-\xi)}{2a^2}\right)\right\}d\xi, \quad t\ge 0,\quad x\ge 0,\quad x-at\le 0,$$

that represent the solutions u_n, where the uniform convergence allows the integral and limit signs to be rearranged.

In view of the above, we have the following theorem.

T h e o r e m 3. *Under conditions* (51), *problem* (1)–(3) *has a unique generalized solution* $u\in C(\overline{Q})$.

P r o o f. The existence has been proven previously. We can write $u\in C\left(\overline{\tilde{Q}^{(1)}}\right)$ and $u\in C\left(\overline{\tilde{Q}^{(2)}}\right)$ using formulas (24) and (51) and the property of the absolute continuity of the Lebesgue integral. Relation (27), which is a consequence of (24) and (51), yields $u\in C(\overline{Q})$.

We note that the generalized solution constructed in Theorem 3 does not depend on the behavior of the function μ on sets of zero measure. Consequently, different values of the number μ_1 yield the same generalized solution.

Under conditions (4), the generalized solution and the classical solution with conjugation conditions (27) and (28) coincide, since both are given by the same formulas and are unique. Recall that the classical solution with conjugation conditions does not depend on the quantity μ_1.

For the same reasons, under assumptions (4) and (26), the solution in the sense of Definition 1 and the generalized solution coincide, since both are given by the same formulas and are unique.

R e m a r k. In the same way a generalized solution can be defined for the data satisfying

$$f\in C^1(\overline{Q}),\quad \varphi\in L^1_{\mathrm{loc}}([0,\infty)),\quad \psi_2\in L^1_{\mathrm{loc}}([0,\infty)),\quad \mu_2\in W^{-1,1}_{\mathrm{loc}}([0,\infty)).$$

Conclusions. In this paper, we have developed a new method that constructs the solution as a limit of classical solutions of auxiliary mixed problems. This approach overcomes the difficulties inherent in earlier methods based on auxiliary problems with conjugation conditions. We have derived necessary and sufficient matching conditions for the existence and uniqueness of the solution. It is shown that the constructed solution satisfies conjugation conditions on the characteristic, reflecting physical conservation laws. The solution is also interpreted as a classical solution of the wave equation with conjugation conditions. Furthermore, we have shown that the problem admits a generalized solution. Under suitable smoothness and matching conditions, the various definitions of the solution are equivalent.

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References

1. Kil'chevskii N. A. *Theory of Collisions between Solid Bodies*. Kiev, Naukova Dumka, 1969. 246 p. (in Russian).

2. Manzhosov V. K. *Longitudinal Impact Models*. Ulyanovsk, Ulyanovsk State Technical University, 2006. 160 p. (in Russian).

3. Gaiduk S. I. Some problems related to the theory of longitudinal impact on a rod. *Differential Equations*, 1977, vol. 13, no. 7, pp. 1233–1243 (in Russian).

4. Moiseev E. I., Kholomeyeva A. A., Frolov A. A. Boundary displacement control for the oscillation process with boundary conditions of damping type for a time less than critical. *Journal of Mathematical Sciences*, 2021, vol. 257, no. 1, pp. 74–84. https://doi.org/10.1007/s10958-021-05471-7

5. Korzyuk V. I., Kozlovskaya I. S., Naumavets S. N. Classical solution of a problem with integral conditions of the second kind for the one-dimensional wave equation. *Differential Equations*, 2019, vol. 55, no. 3, pp. 353–362. https://doi.org/10.1134/s0012266119030091

6. Stolyarchuk I. I. Classical solution of the mixed problem for the string oscillation equation with linear differential polynomials in boundary conditions. *Vestsi Natsyyanal'nai akademii navuk Belarusi. Seryya fizika-matematychnykh navuk = Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series*, 2025, vol. 61, no. 4, pp. 288–298 (in Russian). https://doi.org/10.29235/1561-2430-2025-61-4-288-298

7. Cheb E. S., Siminskaya E. S. Classical solution of the mixed problem for linear nonstrictly hyperbolic fourth-order with multiple characteristics. *Prikladnaya matematika & fizika = Applied Mathematics & Physics*, 2020, vol. 52, no. 1, pp. 11–17 (in Russian).

8. Yatsuk T. A., Basik A. I. On the solvability of a mixed problem for a second order hyperbolic equation in the plane (the case $\lambda_1 < \lambda_1 < 0$). *Formirovanie gotovnosti budushchego uchitelya matematiki k rabote s odarennymi uchashchimisya: sbornik materialov mezhdunarodnoi nauchno-prakticheskoi konferentsii, Brest, 14–15 aprelya 2021 goda* [Developing Readiness in Future Mathematics Teacher to Work with Gifted Students: Collection of Materials of the International Scientific and Practical Conference (Brest, April 14–15, 2021)]. Brest, 2021, pp. 212–216 (in Russian).

9. Yatsuk T. A., Basik A. I. On the solvability of a mixed problem for a second order hyperbolic equation in the plane (the case $\lambda_1 > \lambda_1 > 0$). *Formirovanie gotovnosti budushchego uchitelya matematiki k rabote s odarennymi uchashchimisya: sbornik materialov mezhdunarodnoi nauchno-prakticheskoi konferentsii, Brest, 14–15 aprelya 2021 goda* [Developing Readiness in Future Mathematics Teacher to Work with Gifted Students: Collection of Materials of the International Scientific and Practical Conference (Brest, April 14–15, 2021)]. Brest, 2021, pp. 216–220 (in Russian).

10. Anikonov D. S., Konovalova D. S. Generalized d'Alembert formula for the wave equation with discontinuous coefficients. *Differential Equations*, 2019, vol. 55, no. 2, pp. 270–273. https://doi.org/10.1134/s0012266119020113

11. Korzyuk V. I., Puzyrnyi S. I. Classical solution of mixed problems for the one-dimensional wave equation with the nonsmooth Cauchy conditions. *Vestsi Natsyyanal'nai akademii navuk Belarusi. Seryya fizika-matematychnykh navuk = Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series*, 2016, no. 2, pp. 22–31 (in Russian).

12. Korzyuk V. I., Rudzko J. V. Classical and mild solutions of the Cauchy problem for a mildly quasilinear wave equation with discontinuous and distributional initial conditions. *Journal of Mathematical Sciences*, 2024, vol. 286, no. 4, pp. 535–559. https://doi.org/10.1007/s10958-024-07526-x

13. Li C. G., Li M., Piskarev S., Meerschaert M. M. The fractional d'Alembert's formulas. *Journal of Functional Analysis*, 2019, vol. 277, no. 12, art. ID 108279. https://doi.org/10.1016/j.jfa.2019.108279

14. Ghosh U., Ali M. R., Raut S., Sarkar S., Das S. *D'Alembert's solution of fractional wave equations using complex fractional transformation*. *Arxiv* [Preprint], 2017. Available at: https://arxiv.org/abs/1712.07334. https://doi.org/10.48550/arXiv.1712.07334

15. Koshlyakov N. S., Smirnov M. M., Gliner E. B. *Differential Equations of Mathematical Physics*. Amsterdam, North-Holland Publishing Company, 1964. Pp. xvi, 701. 120 s.

16. Korzyuk V. I., Rudzko J. V., Kolyachko V. V. Solutions of problems with discontinuous conditions for the wave equation. *Zhurnal Belorusskogo gosudarstvennogo universiteta. Matematika. Informatika = Journal of the Belarusian State University. Mathematics and Informatics*, 2023, vol. 3, pp. 6–18 (in Russian).

17. Korzyuk V. I. *Equations of Mathematical Physics*. Moscow, URSS Publ., 2021. 480 p. (in Russian).

18. Korzyuk V. I., Kozlovskaya I. S., Sokolovich V. Yu. Solution of the wave equation in a quarter plane. *Trudy Instituta matematiki = Proceedings of the Institute of Mathematics*, 2020, vol. 28, no. 1–2, pp. 40–56 (in Russian).

19. Polyanin A. D., Nazaikinskii V. E. *Handbook of Linear Partial Differential Equations for Engineers and Scientists*. 2nd ed. New York, Chapman and Hall/CRC, 2015. 1643 p. https://doi.org/10.1201/b19056

20. Korzyuk V. I., Kozlovskaya I. S., Sokolovich V. Yu. Classical solution of the mixed problem in the quarter of the plane for the wave equation. *Doklady Natsional'noi akademii nauk Belarusi = Doklady of the National Academy of Sciences of Belarus*, 2018, vol. 62, no. 6, pp. 647–651 (in Russian). https://doi.org/10.29235/1561-8323-2018-62-6-647-651

21. Rabotnov Yu. N. *Mechanics of Deformable Solid*. Moscow, Nauka Publ., 1979. 744 p. (in Russian).

22. Evans L. C. *Partial Differential Equations*. Providence, American Mathematical Soc., 2010. 749 p.

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